5.6.a. Spontaneous Broken Symmetry for Cooper Pairs

BCS superconductor:

Effective interaction between electrons of opposite spins caused by e-phonon interaction.

→ Bosonic Cooper pairs

Condensate has energy gap

→ superconductivity.

Model Hamiltonian for Cooper pairs with spontaneous broken symmetry (see 4.5._SchrodingerField-.pdf):

$$\mathcal{H} = \frac{\hbar^2}{2M} \nabla\phi^+ \cdot \nabla\phi + \frac{1}{2} g \left(\phi^+ \phi - v^2\right)^2$$

where $M = 2\, m_e$ is the effective mass of the Cooper pairs.

Reminder: g here is the strength of the $\left(\phi^+ \phi\right)^2$ term, i.e., it describes interactions between Cooper pairs, NOT interactions between electrons.

The corresponding Lagrangian (alternative form) is

$$\mathcal{L} = \frac{1}{2} i\hbar\left(\phi^+ \dot{\phi} - \dot{\phi}^+ \phi\right) - \frac{\hbar^2}{2M} \nabla\phi^+ \cdot \nabla\phi - \frac{1}{2} g \left(\phi^+ \phi - v^2\right)^2$$

In the presence of EM fields, minimal coupling

$$p_\mu \to p_\mu - \frac{q}{c} A_\mu \qquad\qquad \partial_\mu \to D_\mu = \partial_\mu + i\frac{q}{\hbar c} A_\mu$$

$$A_\mu = (\varphi,\ -\boldsymbol{A})$$

gives

$$\mathcal{H} = \frac{\hbar^2}{2M}\left(\nabla + i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi^+ \cdot \left(\nabla - i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi + q\,\varphi\,\phi^+ \phi$$

$$+ \frac{1}{2} g \left(\phi^+ \phi - v^2\right)^2 + \mathcal{H}_{\rm EM}$$

$$\mathcal{H}_{\rm EM} = \frac{1}{8\pi}\left(\epsilon\,\boldsymbol{E}^2 + \frac{1}{\mu_m}\boldsymbol{B}^2\right)$$

where $q = 2\,e < 0$ is the effective charge of the Cooper pairs.

$$\mathcal{L} = \frac{1}{2} i\hbar\left[\phi^+\left(\frac{\partial}{\partial t} + i\frac{q}{\hbar c}\varphi\right)\phi - \left(\frac{\partial}{\partial t} - i\frac{q}{\hbar c}\varphi\right)\phi^+ \cdot \phi\right]$$

$$- \frac{\hbar^2}{2M}\left(\nabla + i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi^+ \cdot \left(\nabla - i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi$$

$$- \frac{1}{2} g \left(\phi^+ \phi - v^2\right)^2 + \mathcal{L}_{\rm EM}$$

$$\mathcal{L}_{\rm EM} = -\frac{1}{16\pi}\left(2\,\varepsilon\,F_{0i}\,F^{0i} + \frac{1}{\mu_m} F_{ij}\,F^{ij}\right)$$

Euler eq. for A_0 (see 5.1._MaxwellEquations.pdf)

$$\frac{\partial\mathcal{L}}{\partial\,\partial_i A_0} = -\frac{1}{4\pi}\varepsilon\,F^{i0}$$

$$\frac{\partial\mathcal{L}}{\partial A_0} = \frac{1}{2} i\hbar\left[\phi^+\left(i\frac{q}{\hbar c}\right)\phi - \left(-i\frac{q}{\hbar c}\right)\dot{\phi}^+ \cdot \phi\right] \qquad = -\frac{q}{c}\phi^+ \phi$$

$$\rightarrow \qquad \varepsilon\, \partial_i F^{i\,0} = \frac{4\pi}{c}\, q\, \phi^+ \phi$$

i.e., $\nabla \cdot \boldsymbol{D} = \frac{4\pi}{c}\rho \qquad \rho = q\,\phi^+ \phi$

Euler eq. for $\boldsymbol{A}$:

$$\frac{\partial \mathcal{L}}{\partial\, \partial_i A_j} = -\frac{1}{4\pi}\frac{1}{\mu_m} F^{i\,j} \qquad \frac{\partial \mathcal{L}}{\partial\, \partial_0 A_j} = -\frac{1}{4\pi}\varepsilon F^{0\,j}$$

$$\frac{\partial \mathcal{L}}{\partial A_j} = -\frac{\hbar^2}{2M}\left[-i\frac{q}{\hbar c}\phi^+\left(\partial_j - i\frac{q}{\hbar c}\boldsymbol{A}_j\right)\phi + \left(\partial_j + i\frac{q}{\hbar c}\boldsymbol{A}_j\right)\phi^+ \cdot \left(i\frac{q}{\hbar c}\right)\phi\right]$$

$$= i\frac{q\hbar}{2Mc}\left[\phi^+\left(\partial_j - i\frac{q}{\hbar c}\boldsymbol{A}_j\right)\phi - \left(\partial_j + i\frac{q}{\hbar c}\boldsymbol{A}_j\right)\phi^+ \cdot \phi\right]$$

$$= i\frac{q\hbar}{2Mc}\left[\phi^+ D_j\phi - (D_j\phi)^+\phi\right]$$

$$\rightarrow \qquad \varepsilon\,\partial_0 F^{0\,j} + \frac{1}{\mu_m}\partial_i F^{i\,j} = \frac{4\pi}{c}\frac{q\hbar}{2Mi}\left[\phi^+ D_j\phi - (D_j\phi)^+\phi\right]$$

i.e., $-\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \nabla\times\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}$

where
$$\boldsymbol{J} = \frac{q\hbar}{2Mi}\left[\phi^+\left(\nabla - i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi - \left(\nabla + i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi^+ \cdot \phi\right]$$

$$= \frac{q\hbar}{2Mi}\left[\phi^+\nabla\phi - \left(\nabla\phi^+\right)\phi - 2i\frac{q}{\hbar c}\boldsymbol{A}\phi^+\phi\right]$$

$$= \frac{q\hbar}{2Mi}\left[\phi^+\nabla\phi - \left(\nabla\phi^+\right)\phi\right] - \frac{q^2}{Mc}\boldsymbol{A}\phi^+\phi$$

$$= q\,\boldsymbol{J}_\phi - \frac{q^2}{Mc}\boldsymbol{A}\phi^+\phi$$

where

$$\boldsymbol{J}_\phi = \frac{\hbar}{2Mi}\left[\phi^+\nabla\phi - \left(\nabla\phi^+\right)\phi\right]$$

is the probability current density of the free ϕ field.

The rest of §5.6 will treat ϕ as a classical field.