5.6.c. Meissner Effect

Near the interface of a condensate & vacuum, the interface can be approximated by a plane so that the problem becomes quasi-1-D. Let $\hat{\boldsymbol{n}}$ be the normal to the interface & pointing into the condensate. Then for small enough $z = \boldsymbol{r} \cdot \hat{\boldsymbol{n}}$, we have

$$\boldsymbol{B} = \begin{cases} \boldsymbol{B}(z) & z \geq 0 \ \text{(inside condensate)} \\ \boldsymbol{B}_0 & z < 0 \ \text{(vacuum)} \end{cases}$$

Thus,

$$\frac{d^2 \boldsymbol{B}}{d z^2} = \frac{1}{\lambda^2} \boldsymbol{B}$$

with boundary conditions

$$\boldsymbol{B}(0) \cdot \hat{\boldsymbol{n}} = \boldsymbol{B}_0 \cdot \hat{\boldsymbol{n}} \qquad \hat{\boldsymbol{n}} \times \left[\frac{1}{\mu_m} \boldsymbol{B}(0) - \boldsymbol{B}_0 \right] = \frac{4\pi}{c} \boldsymbol{J}_{//}$$

where $\boldsymbol{J}_{//}$ is the current density at & parallel to the interface.

Thus, the normal component of $\boldsymbol{B}$ dies off as

$$\boldsymbol{B}(z) \cdot \hat{\boldsymbol{n}} = \boldsymbol{B}_0 \cdot \hat{\boldsymbol{n}}\, e^{-z/\lambda}$$

Expression for the parallel components are more complicated but the z dependence is the same.

Therefore, deep enough ($>> \lambda$) inside the condensate, $\boldsymbol{B} = 0$ irregardless.
The condensate is therefore a perfect diamagnet.
This is known as the Meissner effect.

Since $$\boldsymbol{J} = -\frac{c}{4\pi \mu_m \lambda^2} \boldsymbol{A}$$

we have $$\nabla^2 \boldsymbol{J} = \frac{1}{\lambda^2} \boldsymbol{J}$$

as well as $$\nabla \times \boldsymbol{J} = -\frac{c}{4\pi \mu_m \lambda^2} \boldsymbol{B} = -\frac{c}{4\pi \lambda^2} \boldsymbol{H}$$

Furthermore $$\nabla \times \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{J}$$

Thus, $|\,\boldsymbol{X}\,|$, with $\boldsymbol{X} = \boldsymbol{J}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$, all decay as $e^{-z/\lambda}$ into the condensate.
Assuming this is the dominant spatial dependency, then

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{X}) \approx 0 \quad \text{if} \quad \boldsymbol{X} \cdot \hat{\boldsymbol{n}} = 0$$
$$\nabla \times \boldsymbol{X} \approx 0 \quad \text{if} \quad \boldsymbol{X} // \hat{\boldsymbol{n}}$$

In words, $\nabla \times \boldsymbol{X}$ is parallel to the interface if $\boldsymbol{X}$ is.

In the special case $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{n}}$, the normal boundary condition gives

$$\boldsymbol{B}(0) \cdot \hat{\boldsymbol{n}} = B_0$$

Consider the ansatz

$$\boldsymbol{B}(z) = B(z)\, \hat{\boldsymbol{n}} = B_0\, e^{-z/\lambda}\, \hat{\boldsymbol{n}}$$

so that

$$\boldsymbol{J} = \frac{c}{4\pi \mu_m} \nabla \times \boldsymbol{B} = \frac{c}{4\pi \mu_m} B_0 \left(\nabla e^{-z/\lambda} \right) \times \hat{\boldsymbol{n}} \propto \hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}} = 0$$

It is then trivial to verify that the ansatz also satisfies all the other relations.

Thus, an external field normal to the interface is continuous across the interface & dies exponentially inside the condensate. Furthermore, no current is induced.

For an exteral field parallel to the interface, i.e.,

$$\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{b}} \qquad \text{with} \qquad \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}} = 0$$

we have

$$\boldsymbol{B}(0)\cdot\hat{\boldsymbol{n}} = 0$$

Consider the ansatz

$$\boldsymbol{B}(z) = B(z)\,\hat{\boldsymbol{b}} = B_0\, e^{-z/\lambda}\,\hat{\boldsymbol{b}}$$

so that

$$\begin{aligned}\boldsymbol{J} &= \frac{c}{4\,\pi\,\mu_m}\,\nabla\times\boldsymbol{B} = \frac{c}{4\,\pi\,\mu_m}\,B_0\left(\nabla e^{-z/\lambda}\right)\times\hat{\boldsymbol{b}}\\ &= -\frac{c}{4\,\pi\,\mu_m\,\lambda}\,B_0\,e^{-z/\lambda}\,\hat{\boldsymbol{n}}\times\hat{\boldsymbol{b}}\\ &= -\frac{c}{4\,\pi\,\mu_m\,\lambda}\,\hat{\boldsymbol{n}}\times\boldsymbol{B}(z) = -\frac{c}{4\,\pi\,\lambda}\,\hat{\boldsymbol{n}}\times\boldsymbol{H}(z)\end{aligned}$$

Thus the ansatz doesn’t satisfies the tangential boundary condition.
The obvious remedy is

$$\boldsymbol{B}(z) = e^{-z/\lambda}\left(B_1\,\hat{\boldsymbol{b}} + B_2\,\hat{\boldsymbol{n}}\times\hat{\boldsymbol{b}}\right)$$

which still leaves $\boldsymbol{J}$ lying parallel to the interface.

Thus, an external field parallel to the interface is discontinuous across the interface & dies exponentially inside the condensate. Furthermore, a surface current is induced.