5.6.d. Flux Quantization

Consider an infinite superconductor with a cylindrical hole of radius R and axis parrallel to an external magnetic field.

Let the cylinder axis be the z-axis and set (c.f. §5.4)

$$\phi(x) = v\, e^{i\chi(x)} \qquad\qquad A_\mu(x) = U_\mu(x) - \frac{\hbar c}{q}\partial_\mu \chi(x)$$

In the absence of electric field, we can set

$$A_0 = U_0 = \partial_0 \chi = 0$$

& $$\boldsymbol{A} = \boldsymbol{U} + \frac{\hbar c}{q}\nabla\chi$$

For $r > R$ (inside the superconductor) :

$$\mathcal{H} = \frac{\hbar^2}{2M}\left(\nabla + i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi^* \cdot \left(\nabla - i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi + q\,\varphi\,\phi^*\phi$$
$$+\frac{1}{2}g\left(\phi^*\phi - v^2\right)^2 + \mathcal{H}_{\text{EM}}$$

with $$\mathcal{H}_{\text{EM}} = \frac{1}{8\pi}\left(\epsilon\,\boldsymbol{E}^2 + \frac{1}{\mu_m}\boldsymbol{B}^2\right)$$

Using

$$\left(\nabla + i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi^* = i\frac{q}{\hbar c}\boldsymbol{U}\,\phi^*$$
$$\left(\nabla - i\frac{q}{\hbar c}\boldsymbol{A}\right)\phi = -i\frac{q}{\hbar c}\boldsymbol{U}\,\phi$$
$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{U}$$

we have

$$\mathcal{H} = \frac{q^2 v^2}{2Mc^2}\boldsymbol{U}^2 + \frac{1}{8\pi\mu_m}\boldsymbol{B}^2$$

$$\lambda = \sqrt{\frac{M}{4\pi\mu_m}}\,\frac{c}{q\,v}$$

→ $$\mathcal{H} = \frac{1}{8\pi\mu_m\lambda^2}\boldsymbol{U}^2 + \frac{1}{8\pi\mu_m}\boldsymbol{B}^2$$
$$= \frac{1}{8\pi\mu_m}\left(\frac{1}{\lambda^2}\boldsymbol{U}^2 + (\nabla\times\boldsymbol{U})^2\right)$$

Since $|\,\boldsymbol{A}\,|$ decays exponentially into the condensate, we set

$$\boldsymbol{U}(x) = 0 \qquad \text{as} \qquad r\to\infty$$

Actually, the problem as posed is time independent.

From §5.6.a, we have

$$\boldsymbol{J} = \frac{q\hbar}{2Mi}\left[\phi^*\nabla\phi - (\nabla\phi^*)\,\phi\right] - \frac{q^2}{Mc}\boldsymbol{A}\,\phi^*\phi$$

Using

$$\phi^*\nabla\phi = i\,v^2\nabla\chi \qquad\qquad (\nabla\phi^*)\,\phi = -i\,v^2\nabla\chi$$

we have

$$\boldsymbol{J} = \frac{q\hbar v^2}{M}\nabla\chi - \frac{q^2 v^2}{M c}\left(\boldsymbol{U} + \frac{\hbar c}{q}\nabla\chi\right)$$
$$= -\frac{q^2 v^2}{M c}\boldsymbol{U} \ = -\frac{c}{4\pi\mu_m\lambda^2}\boldsymbol{U}$$

Comparing with the London eq.

$$\boldsymbol{J} = -\frac{c}{4\pi\mu_m\lambda^2}\boldsymbol{A}$$

we se that $\boldsymbol{U}$ has taken on the role of $\boldsymbol{A}$ so that

$$\nabla^2\boldsymbol{U} = \frac{1}{\lambda^2}\boldsymbol{U}$$
$$\nabla^2\boldsymbol{B} = \frac{1}{\lambda^2}\boldsymbol{B}$$

Since

$$\nabla\times\boldsymbol{B} = \nabla\times(\nabla\times\boldsymbol{U}) \quad = \nabla(\nabla\cdot\boldsymbol{U}) - \nabla^2\boldsymbol{U}$$
$$= -\nabla^2\boldsymbol{U} \qquad \text{(Coumlomb gauge)}$$
$$= -\frac{1}{\lambda^2}\boldsymbol{U}$$
$$= \frac{4\pi\mu_m}{c}\boldsymbol{J}$$

which is just the Ampere’s law.

Assuming the exponent decay along the radial direction is the dominant spatial variation of all quantities, $\nabla\times\boldsymbol{B}$ has no significant radial component so that $\boldsymbol{J}$ must lie on the cylindrical surface of the hole. By symmetry, it must be a circulation around the hole.

The total flux through the xy plane is

$$\Phi = \int_0^{R'>>\mathrm{R}} r\,dr\int_0^{2\pi} d\theta\, B_z$$
$$= \oint_C d\boldsymbol{r}\cdot\boldsymbol{A}$$

where C is a circle of radius $R' >> \mathrm{R}$ & centered at the origin.

Whereupon, $\boldsymbol{U} = 0$ so that $\boldsymbol{A} = \frac{\hbar c}{q}\nabla\chi$.

Hence, $$\Phi = \frac{\hbar c}{q}\int_0^{2\pi} R'\,d\theta\,\frac{d\chi}{R'\,d\theta}$$
$$= \frac{\hbar c}{q}\oint_C d\chi$$
$$= \frac{\hbar c}{q}\Delta\chi$$

where $\Delta\chi$ is the change of χ in going around the circle. From the definition $\phi = v\,e^{i\chi}$, we see that ϕ is single-valued (as it must be) only if

$$\Delta\chi = 2\pi n \qquad \text{where} \quad n = 0, \pm1, \pm2, \ldots$$

Taking Φ to be positive, we have

$$\Phi = \frac{2\pi\hbar c}{|\,q\,|}n \qquad n = 0, 1, 2, \ldots$$
$$= n\,\Phi_L$$

i.e., Φ is quantized in unit of the London flux

$$\Phi_L = \frac{2\pi\hbar c}{|q|} = \frac{h c}{|q|} = \frac{h c}{|2 e|}$$

where e is the electron charge. Incidentally, the Dirac flux is defined as

$$\Phi_D = \frac{h c}{|e|}$$