5.6.e. Vortices

Let's treat the hole problem, with $R \to 0$, more rigorously. Starting with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

the ground state will be θ-independent so

$$\nabla^2 B_z = \frac{1}{\lambda^2} B_z$$

$$\to \qquad \left(\frac{d^2}{d r^2} + \frac{1}{r}\frac{d}{d r} \right) B_z = \frac{1}{\lambda^2} B_z$$

Solutions to this are modified Bessel functions I_ν & K_ν (see G.B.Arfken et al, "Mathematical Methods for Physicists", 7th ed., §14.5).

Formally, a solution is splitted in two parts, I_ν for $r < R$ & K_ν for $r > R$, to keep it regular (finite) at $r = 0$ & ∞, respectively. However, for $R \to 0$, we need only deal with $r > R$.

For a solution that dies off as $r \to \infty$, we have ,

$$B_z(r) = \alpha K_0\left(\frac{r}{\lambda}\right)$$

where α is a constant & K_0 is the modified Bessel function with

$$K_0(x) \to \begin{cases} -(\ln x + \gamma - \ln 2) & x \to 0 \\ \sqrt{\frac{\pi}{2x}}\, e^{-x} & x \to \infty \end{cases}$$

and γ is the Euler-Mascheroni constant.

The mild singularity $\ln x$ as $x \to 0$ is the price to pay for a non-zero flux within the hole.

From the definition $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we see that $\boldsymbol{A}$ must be differentiable everywhere. Since $\boldsymbol{A} = \boldsymbol{U} + \frac{\hbar c}{q} \nabla \chi$ (see §5.6.d), we have $\boldsymbol{B} = \nabla \times \boldsymbol{U}$ wherever $\boldsymbol{U}$ is differentiable. However, wherever $\boldsymbol{U}$ is singular, we must bring back the χ term to make sure $\boldsymbol{A}$ is still differentiable.

$$\boldsymbol{U} = -\lambda^2 \nabla \times \boldsymbol{B}$$

$$\to \qquad \boldsymbol{U}_i = -\lambda^2 \varepsilon_{ijk} \partial_j \boldsymbol{B}_k = -\lambda^2 \varepsilon_{ij3} \partial_j B_z$$

$$= -\alpha \lambda \varepsilon_{ij3} \frac{\partial r}{\partial x^j} K_0'\left(\frac{r}{\lambda}\right)$$

$$r = \sqrt{-x^j x_j} \qquad \to \qquad \frac{\partial r}{\partial x^j} = \frac{x^j}{r}$$

$$\therefore \qquad \boldsymbol{U}_i = -\alpha \lambda \varepsilon_{ij3} \frac{x^j}{r} K_0'\left(\frac{r}{\lambda}\right)$$

From the recurrence

$$K_{\nu-1} + K_{\nu+1} = -2 K_\nu' \qquad\qquad K_{-1} = K_1$$

we have

$$K_0' = -K_1$$

$$\to \qquad \boldsymbol{U}_i = \alpha \lambda K_1\left(\frac{r}{\lambda}\right) \varepsilon_{ij3} \frac{x^j}{r}$$

For $x \to 0$,

$$K_\nu(x) = 2^{\nu-1} (\nu-1)!\, x^{-\nu} + \dots$$

$$\rightarrow \qquad \boldsymbol{U}_i \approx \alpha\, \lambda^2\ \varepsilon_{i\,j\,3}\, \frac{x^j}{r^2}$$

which is singular at $r = 0$.

Thus,

$$\boldsymbol{A}_i = \boldsymbol{U}_i + \frac{\hbar\, c}{q}\, \partial_i\, \chi$$

$$\approx \alpha\, \lambda^2\ \varepsilon_{i\,j\,3}\, \frac{x^j}{r^2} + \frac{\hbar\, c}{q}\, \partial_i\, \chi \qquad \text{as } r \to 0$$

From the single-valuedness condition on $\phi = v\, e^{i\,\chi}$, i.e.,

$$\Delta\chi = 2\,\pi\, n$$

we see that

$$\chi = n\,\theta \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

where θ is the azimuthal angle.

$$\rightarrow \qquad \partial_i\, \chi = n\, \partial_i\, \theta\ = n\, \partial_i\ \tan^{-1}\frac{x^2}{x^1}\ = n\, \frac{\left(x^1\right)^2}{r^2}\, \partial_i\, \frac{x^2}{x^1}$$

$$\therefore \qquad \partial_1\, \chi = -n\, \frac{x^2}{r^2} \qquad \partial_2\, \chi = n\, \frac{x^1}{r^2}$$

or

$$\partial_i\, \chi = -n\, \varepsilon_{i\,j\,3}\, \frac{x^j}{r^2}$$

$$\rightarrow \qquad \boldsymbol{A}_i \approx \varepsilon_{i\,j\,3}\, \frac{x^j}{r^2}\left(\alpha\, \lambda^2\ - n\, \frac{\hbar\, c}{q}\right) \qquad \text{as } r \to 0$$

To remove the singularity, we set

$$\alpha\, \lambda^2\ - n\, \frac{\hbar\, c}{q} = 0$$

$$\rightarrow \qquad \alpha = n\, \frac{\hbar\, c}{q\, \lambda^2}\ = n\, \frac{\Phi_L}{2\,\pi\, \lambda^2} \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

where

$$\Phi_L = \frac{2\,\pi\, \hbar\, c}{|\ q\ |} = \frac{h\, c}{|\ q\ |}$$

Hence,

$$B_z(r) = n\, \frac{\Phi_L}{2\,\pi\, \lambda^2}\, K_0\!\left(\frac{r}{\lambda}\right)$$

Using (M.Abramowitz, I.A.Stegan,"Handbook of Mathematical Functions", formula 11.4.22)

$$\int_0^\infty t^\mu\, K_v(x)\, d\, x = 2^{\mu-1}\, \Gamma\!\left(\frac{\mu + v + 1}{2}\right) \Gamma\!\left(\frac{\mu - v + 1}{2}\right)$$

we have

$$\int_0^\infty x\, K_0(x)\, d\, x = \Gamma(1)\, \Gamma(1) = 1$$

Hence, the flux associatied with the hole is

$$\Phi = 2\,\pi \int_0^\infty B_z(r)\, r\, d\, r$$

$$= n\, \Phi_L \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

as it should be.