5.6.f. Energy of a Vortex

$$\boldsymbol{U} = -\lambda^2 \nabla \times \boldsymbol{B}$$

$$\rightarrow \quad \mathcal{H} = \frac{1}{8\pi\mu_m}\left(\frac{1}{\lambda^2}\boldsymbol{U}^2 + \boldsymbol{B}^2\right)$$

$$= \frac{1}{8\pi\mu_m}\left[\lambda^2 (\nabla \times \boldsymbol{B})^2 + \boldsymbol{B}^2\right]$$

$$\boldsymbol{B} = B_z(r)\,\hat{\boldsymbol{z}}$$

$$\rightarrow \quad \boldsymbol{B}^2 = B_z^2$$

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_z \end{vmatrix} = (\partial_y B_z, -\partial_x B_z, 0)$$

$$\therefore \quad H = \int d^2 r\, \mathcal{H}$$

$$= \frac{1}{8\pi\mu_m}\int d^2 r\left[\lambda^2 (\partial_y B_z)^2 + \lambda^2 (\partial_x B_z)^2 + B_z^2\right]$$

$$\int d^2 r\,(\partial_y B_z)^2 = \int d^2 r\left[\partial_y (B_z \partial_y B_z) - B_z \partial_y^2 B_z\right]$$

$$\int d^2 r\,(\partial_x B_z)^2 = \int d^2 r\left[\partial_x (B_z \partial_x B_z) - B_z \partial_x^2 B_z\right]$$

$$\rightarrow \quad \int d^2 r\left[(\partial_y B_z)^2 + (\partial_x B_z)^2\right]$$

$$= \int d^2 r\,\nabla\cdot(B_z \nabla B_z) - \int d^2 r\, B_z \nabla^2 B_z$$

where ∇ is the 2-D gradient in the xy plane.

Using the 2-D version of the Gauss theorem, we have

$$\int d^2 r\,\nabla\cdot(B_z \nabla B_z) = \oint_C d\boldsymbol{r}\cdot B_z \nabla B_z$$

where C goes counterclockwise around the boundary of the integration area in which the integrand is analytic.

Finally, with $\nabla^2 \boldsymbol{B} = \frac{1}{\lambda^2}\boldsymbol{B}$, we get

$$H = \frac{\lambda^2}{8\pi\mu_m}\oint_C d\boldsymbol{r}\cdot B_z \nabla B_z$$

Since B_z has a single singularity at $r = 0$, we have

$$C = c_\infty + l_{in} + l_{out} - c_0$$

where c_r is a counterclockwise circle around the origin with radius r. l_{in} & l_{out} are two adjacent lines joining the two circles. Contribution from c_∞ vanishes since $B_z \to 0$ as $r \to \infty$. Contributions from l_{in} & l_{out} cancel each other.

Hence,

$$H = -\frac{\lambda^2}{8\pi\mu_m}\oint_{c_0} d\boldsymbol{r}\cdot B_z \nabla B_z$$

Using

$$B_z \simeq -n\frac{\Phi_L}{2\pi\lambda^2}\ln\frac{r}{\lambda}$$

$$\rightarrow \qquad \nabla B_z \simeq -n \frac{\Phi_L}{2\pi\lambda^2} \frac{1}{r} \hat{\boldsymbol{r}}$$

$$H = -\frac{\lambda^2}{8\pi\mu_m} \left(\frac{n\Phi_L}{2\pi\lambda^2}\right)^2 \lim_{r\to 0} \int_0^{2\pi} r\, d\theta \frac{1}{r} \ln\frac{r}{\lambda}$$

$$= \frac{1}{4\mu_m} \left(\frac{n\Phi_L}{2\pi\lambda}\right)^2 \ln\frac{\lambda}{\xi}$$

where ξ is a small cut-off radius introduced by hand to prevent $H \to \infty$.

On the other hand, one can trace the origin of the singularity back to the assumption

$$\phi(x) = v\, e^{i\chi} = v\, e^{i n\theta} \qquad v = \text{const}$$

since θ is not defined at $r = 0$.

A remedy to this is to set

$$\phi(x) = v(r)\, e^{i n\theta} \text{with } v(0) = 0$$

which mean $r = 0$ is not part of the condensate (i.e., non-superconducting). Physically, this can be achieved by having the hole radius $R > \xi$, where ξ is the coherence length. In other words, ξ is just the cut-off radius introduced earlier.

Note that

$$H \text{ is } \begin{cases} < 0 & \text{if } \xi > \lambda \\ > 0 & \text{if } \xi < \lambda \end{cases}$$

which is the major cause for the existence of 2 types, I & II, of superconductors.

Assuming $\xi < \lambda$, creation of a vortex of flux $n\Phi_L$ requires energy $H \propto (n\Phi_L)^2$. However, the same total flux can be accommodated by n vortices each of flux Φ_L for a total cost of energy $H \propto n\ \Phi_L^2$. Thus, vortices in type II superconductors tend to carry flux $\pm\Phi_L$.