5.6.g. Interaction between Vortices

Consider 2 vortices located at $\boldsymbol{r}_1$ & $\boldsymbol{r}_2$.

$$B_z(r) = n\,\frac{\Phi_L}{2\pi\lambda^2} K_0\!\left(\frac{r}{\lambda}\right) \quad \text{for 1 vortex at origin}$$

$$\rightarrow \quad B_z(r) = \frac{\Phi_L}{2\pi\lambda^2}\left[n_1 K_0\!\left(\frac{\rho_1}{\lambda}\right) + n_2 K_0\!\left(\frac{\rho_2}{\lambda}\right)\right]$$

$$= n_1\, b_z(\rho_1) + n_2\, b_z(\rho_2)$$

where

$$b_z(\rho_i) = \frac{\Phi_L}{2\pi\lambda^2} K_0\!\left(\frac{\rho_i}{\lambda}\right) \qquad \rho_i = |\,\boldsymbol{r} - \boldsymbol{r}_i\,|$$

Caution: In this section, we'll use

subscripts i, j, ... to denote vortex identity

& α, β, ... to denote Cartesian components.

$$H = -\frac{\lambda^2}{8\pi\mu_m}\oint_{c_0} d\boldsymbol{r}\cdot B_z \nabla B_z \quad \text{for 1 vortex at origin}$$

$$= \frac{1}{4\mu_m}\left(\frac{n\Phi_L}{2\pi\lambda}\right)^2 \ln\frac{\lambda}{\xi}$$

$$\rightarrow \quad H = -\frac{\lambda^2}{8\pi\mu_m}\oint_{c} d\boldsymbol{r}\cdot B_z \nabla B_z$$

where

$$c = c_1 + l_{12} + l_{21} + c_2 = c_1 + c_2$$

with c_i a counterclockwise infinitesimal circle around $\boldsymbol{r}_i$ & l_{ij} a line going from $\boldsymbol{r}_i$ to $\boldsymbol{r}_j$.

$$(B_1 + B_2)\nabla(B_1 + B_2)$$

$$= B_1\nabla B_1 + B_2\nabla B_2 + B_1\nabla B_2 + B_2\nabla B_1$$

$$= B_1\nabla B_1 + B_2\nabla B_2 + \nabla(B_1 B_2)$$

$$\rightarrow \quad H = \frac{1}{4\mu_m}\left(\frac{\Phi_L}{2\pi\lambda}\right)^2 (n_1^2 + n_2^2)\ln\frac{\lambda}{\xi} + H_{\text{int}}$$

where

$$H_{\text{int}} = -\frac{\lambda^2 n_1 n_2}{8\pi\mu_m}\oint_{c} d\boldsymbol{r}\cdot\nabla[\,b_z(\rho_1)\, b_z(\rho_2)\,]$$

$$= -\frac{\lambda^2 n_1 n_2}{8\pi\mu_m}\left(\frac{\Phi_L}{2\pi\lambda^2}\right)^2 \oint_{c} d\boldsymbol{r}\cdot\nabla\left[K_0\!\left(\frac{\rho_1}{\lambda}\right) K_0\!\left(\frac{\rho_2}{\lambda}\right)\right]$$

On c_1,

$$b_z(\rho_2) \approx b_z(\rho_{12}) \approx \text{constant} \qquad \rho_{ij} = |\,\boldsymbol{r}_i - \boldsymbol{r}_j\,| = \rho_{ji}$$

On c_2,

$$b_z(\rho_1) \approx b_z(\rho_{12}) \approx \text{constant}$$

$$\rightarrow \quad H_{\text{int}} \approx -2\times\frac{\lambda^2 n_1 n_2}{8\pi\mu_m}\left(\frac{\Phi_L}{2\pi\lambda^2}\right)^2 K_0\!\left(\frac{\rho_{12}}{\lambda}\right)\oint_{c_0} d\boldsymbol{r}\cdot\nabla K_0\!\left(\frac{r}{\lambda}\right)$$

$$\int d\boldsymbol{r}\cdot\nabla f = f$$

$$\rightarrow \quad \oint_{c_0} d\boldsymbol{r}\cdot\nabla f = \Delta f$$

where Δf = change of f in going around c_0.

$$\therefore \qquad \oint_{c_0} d\boldsymbol{r}\cdot\nabla K_0\left(\frac{r}{\lambda}\right) = \Delta K_0\left(\frac{r}{\lambda}\right) = -\Delta \ln\frac{r}{\lambda} = -2\pi$$

$$H_{\text{int}} \approx \frac{n_1 n_2}{2\mu_m}\left(\frac{\Phi_L}{2\pi\lambda}\right)^2 K_0\left(\frac{\rho_{12}}{\lambda}\right)$$

$$= \frac{n_1 n_2}{4\pi\mu_m}\frac{\Phi_L^2}{2\pi\lambda^2} K_0\left(\frac{\rho_{12}}{\lambda}\right)$$

Since $K_v(x) > 0 \;\; \forall \; v \; \& \, x$,

$$\text{sgn}(H_{\text{int}}) = \text{sgn}(n_1 n_2)$$

i.e., vortices with parallel fluxes repulse each other,
while those with anti-parallel fluxes attract.

Force on vortex i is

$$\boldsymbol{f}_i = -\frac{\partial}{\partial \boldsymbol{r}_i} H_{\text{int}}$$

Using

$$K_0' = -K_1$$

$$\frac{\partial}{\partial \boldsymbol{r}_1}\rho_{12} = \hat{\boldsymbol{r}}_{21} \qquad\qquad \frac{\partial}{\partial \boldsymbol{r}_2}\rho_{12} = \hat{\boldsymbol{r}}_{12}$$

where $\hat{\boldsymbol{r}}_{ij} = \dfrac{\boldsymbol{r}_j - \boldsymbol{r}_i}{\rho_{ij}}$ = unit vector pointing from $\hat{\boldsymbol{r}}_i$ to $\hat{\boldsymbol{r}}_j$, we have

$$\boldsymbol{f}_1 = \frac{n_1 n_2}{2\mu_m}\left(\frac{\Phi_L}{2\pi\lambda}\right)^2 K_1\left(\frac{\rho_{12}}{\lambda}\right)\frac{1}{\lambda}\hat{\boldsymbol{r}}_{21} = -\boldsymbol{f}_2$$

On the other hand, using

$$b_z(r) = \frac{\Phi_L}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$$

to rewrite

$$H_{\text{int}} = \frac{n_1 n_2}{4\pi\mu_m}\frac{\Phi_L^2}{2\pi\lambda^2} K_0\left(\frac{\rho_{12}}{\lambda}\right)$$

as

$$H_{\text{int}} = \frac{n_1 n_2}{4\pi\mu_m}\Phi_L\, b_z\left(\frac{\rho_{12}}{\lambda}\right)$$

we have

$$\boldsymbol{f}_i = -\frac{n_1 n_2}{4\pi\mu_m}\Phi_L\,\frac{\partial}{\partial \boldsymbol{r}_i} b_z\left(\frac{\rho_{12}}{\lambda}\right)$$

Using $\alpha = 1, 2, 3$ to denote the Cartesian components, we have

$$\boldsymbol{f}_{i\alpha} = -\frac{n_1 n_2}{4\pi\mu_m}\Phi_L\,\frac{\partial}{\partial \boldsymbol{r}_{i\alpha}} b_z\left(\frac{\rho_{12}}{\lambda}\right)$$

Ampere's law $\qquad \nabla\times\boldsymbol{H} = \dfrac{4\pi}{c}\boldsymbol{J}$

$$\to \qquad \boldsymbol{J}_\alpha = \frac{c}{4\pi\mu_m}\varepsilon_{\alpha\beta\gamma}\,\partial_\beta \boldsymbol{B}_\gamma$$

$$= \frac{c}{4\pi\mu_m}\varepsilon_{\alpha\beta 3}\,\partial_\beta b_z \quad \text{if} \quad \boldsymbol{B} = b_z\,\hat{\boldsymbol{z}}$$

$$\therefore \qquad \varepsilon_{\alpha\gamma 3} \boldsymbol{J}_\alpha = \frac{c}{4\pi\mu_m} \varepsilon_{\alpha\gamma 3} \varepsilon_{\alpha\beta 3} \partial_\beta b_z$$

$$= \frac{c}{4\pi\mu_m} \left(\delta_{\gamma\beta} \delta_{33} - \delta_{\gamma 3} \delta_{3\beta} \right) \partial_\beta b_z$$

$$= \frac{c}{4\pi\mu_m} \partial_\gamma b_z$$

or $$\frac{c}{4\pi\mu_m} \partial_\alpha b_z = \varepsilon_{\gamma\alpha 3} \boldsymbol{J}_\gamma = -\varepsilon_{\alpha\beta 3} \boldsymbol{J}_\beta$$

$$\rightarrow \qquad \boldsymbol{f}_{i\,\alpha} = \frac{n_1 n_2}{c} \Phi_L \; \varepsilon_{\alpha\beta 3} \boldsymbol{J}_{i\,\beta}$$

Since $\boldsymbol{f}_i$ is the force on vortex i, which is at $\boldsymbol{r}_i$, $\boldsymbol{J}_i$ is necessarily the current density at $\boldsymbol{r}_i$.