5.7.a. Fields of a Long Solenoid

Consider an infinitely long solenoid carrying a steady current.

Let R be the radius of the solenoid & the z-axis be along the solenoid axis. Then

$$\mathbf{B} = \begin{cases} B\,\hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases}$$

where B is a constant and r is the perpendicular distance to the axis.

The flux through any plane S that cuts across the entire cylinder & parallel to the xy plane is

$$\Phi = \int_S d\mathbf{S}\cdot\mathbf{B} = \pi R^2 B$$

$$= \int d\mathbf{S}\cdot(\nabla\times\mathbf{A}) = \oint_C d\mathbf{r}\cdot\mathbf{A}$$

where C is the boundary of S & hence lies entirely outside the solenoid.

Since $\Phi \neq 0$, $\mathbf{A}$ cannot be identically 0 on C.

Since $\mathbf{A}$ must be continuous everywhere, it does not vanish completely outside the solenoid, where $\mathbf{B} = 0$ everywhere.

In cylindrical coordinates,

$$\nabla\times\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \partial_r & \partial_\theta & \partial_z \\ A_r & r\,A_\theta & A_z \end{vmatrix} = B\,\hat{\mathbf{z}}$$

$\rightarrow \qquad \partial_\theta A_z = r\,\partial_z A_\theta \qquad\qquad \partial_r A_z = \partial_z A_r$

& $$\partial_r (r\,A_\theta) - \partial_\theta A_r = \begin{cases} r\,B & r < R \\ 0 & r > R \end{cases}$$

By symmetry, we can assume everything is independent of z & θ. Hence, we're left with

$$\partial_r A_z = 0$$

$$\partial_r (r\,A_\theta) = \begin{cases} r\,B & r < R \\ 0 & r > R \end{cases}$$

Apart from some additive constants, we thus have

$$A_r = A_z = 0$$

& $$r\,A_\theta = \begin{cases} \frac{1}{2} r^2 B & r < R \\ \text{const} & r > R \end{cases}$$

Continuity of $\mathbf{A}$ then gives

$$A_\theta = \begin{cases} \frac{1}{2} r\,B & r < R \\ \frac{1}{2r} R^2 B = \frac{1}{2\pi r}\Phi & r > R \end{cases}$$