5.7.b. Gauge Invariant Matter Field

The matter (Schrodinger, K-G, ... etc) field $\phi(x)$ for a charged particle is not invariant under the (local) gauge transformation

$$\phi(x) \rightarrow e^{i\Lambda(x)}\phi(x)$$
$$A_\mu \rightarrow A_\mu - \frac{\hbar c}{q}\partial_\mu \Lambda$$

Thus, $\phi(x)$ is not a physical quantity, which must be gauge invariant.

For steady state systems in the absence of $\boldsymbol{E}$, we can set $\varphi = A_0 = 0$ so that the gauge transformation reduces to

$$\phi(\boldsymbol{r}) \rightarrow e^{i\Lambda(\boldsymbol{r})}\phi(\boldsymbol{r})$$
$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \frac{\hbar c}{q}\nabla\Lambda$$

A gauge invariant version of $\phi(\boldsymbol{r})$ is

$$\phi_C(\boldsymbol{r}) = \exp\left(-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\boldsymbol{A}\right)\phi(\boldsymbol{r})$$

where the integration goes from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ along path C.

Thus, under a gauge transformation,

$$\phi_C(\boldsymbol{r}) \rightarrow \exp\left[-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\left(\boldsymbol{A} + \frac{\hbar c}{q}\nabla'\Lambda\right)\right]e^{i\Lambda}\phi(\boldsymbol{r})$$

Using

$$\int_{r_0}^{r} d\boldsymbol{r}'\cdot\nabla'\Lambda = \Lambda(\boldsymbol{r}) - \Lambda(\boldsymbol{r}_0)$$

we have

$$\phi_C(\boldsymbol{r}) \rightarrow e^{i\Lambda(\boldsymbol{r}_0)}\phi_C(\boldsymbol{r})$$

Since $e^{i\Lambda(\boldsymbol{r}_0)}$ is just a constant of magnitude 1, $e^{i\Lambda(\boldsymbol{r}_0)}\phi_C(\boldsymbol{r})$ is physically indistinguishable from $\phi_C(\boldsymbol{r})$.

Using

$$\nabla\exp\left(-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\boldsymbol{A}\right) = -\frac{i}{\hbar c}q\boldsymbol{A}\exp\left(-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\boldsymbol{A}\right)$$

we have

$$\nabla\phi_C(\boldsymbol{r}) = \exp\left(-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\boldsymbol{A}\right)\left(\nabla - \frac{i}{\hbar c}q\boldsymbol{A}\right)\phi(\boldsymbol{r})$$
$$= \exp\left(-\frac{i}{\hbar c}q\int_{r_0}^{r} d\boldsymbol{r}'\cdot\boldsymbol{A}\right)\boldsymbol{D}\phi(\boldsymbol{r})$$

where

$$\boldsymbol{D} = \nabla - \frac{i}{\hbar c}q\boldsymbol{A}$$

is the spatial part of the covariant derivative

$$D_\mu = \partial_\mu + i\frac{q}{\hbar c}A_\mu$$

Caution: Unlike $\boldsymbol{A}$, which is a contravariant vector, $\boldsymbol{D}$ is a covariant vector like ∇.

If C_i, $i = 1, 2$, are 2 paths going from $\boldsymbol{r}_0$ to $\boldsymbol{r}$, then

$$\int_{C_1} - \int_{C_2} = \pm \oint \quad \text{if } C_1 \text{ is on the } \begin{matrix} \text{right} \\ \text{left} \end{matrix} \text{ of } C_2$$

(By convention, $\oint$ is always counterclockwise.)

$$\rightarrow \qquad \phi_{C_1}(\boldsymbol{r}) = \exp\left(\mp \frac{i}{\hbar c} q \oint d\boldsymbol{r}' \cdot \boldsymbol{A} \right) \phi_{C_2}(\boldsymbol{r})$$

$$= \exp\left(\mp \frac{i}{\hbar c} q \Phi \right) \phi_{C_2}(\boldsymbol{r})$$

$\dfrac{q\Phi}{\hbar c}$ is called the Aharonov-Bohm phase.

Hamiltonian

Applying the minimal coupling to a Schrodinger field

$$H = \int d^3 r \left(\frac{\hbar^2}{2m} \nabla\phi^+ \cdot \nabla\phi + V \phi^+ \phi \right)$$

we get

$$H = \int d^3 r \left(\frac{\hbar^2}{2m} \boldsymbol{D}\phi^+ \cdot \boldsymbol{D}\phi + V \phi^+ \phi \right)$$

Using

$$\phi_C^+ \phi_C = \phi^+ \phi$$

$$\& \qquad \nabla\phi_C(\boldsymbol{r}) = \exp\left(-\frac{i}{\hbar c} q \int_{r_0}^{r} d\boldsymbol{r}' \cdot \boldsymbol{A} \right) \boldsymbol{D}\phi(\boldsymbol{r})$$

$$\rightarrow \qquad \nabla\phi_C^+(\boldsymbol{r}) = \exp\left(\frac{i}{\hbar c} q \int_{r_0}^{r} d\boldsymbol{r}' \cdot \boldsymbol{A} \right) [\boldsymbol{D}\phi(\boldsymbol{r})]^+$$

$$\nabla\phi_C^+ \cdot \nabla\phi_C = (\boldsymbol{D}\phi)^+ \boldsymbol{D}\phi$$

$$\therefore \qquad H = \int d^3 r \left(\frac{\hbar^2}{2m} \nabla\phi_C^+ \cdot \nabla\phi_C + V \phi_C^+ \phi_C \right)$$