5.7.d. Simple Dynamic Model

Consider a particle moving in a potential of axial symmetry $V(r)$, r being the perpendicular distance from the z-axis.

In the presence of a magnetic field, the Hamiltonian becomes

$$H = \frac{1}{2M}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 + V(r)$$

If $\boldsymbol{B}$ is confined to the z-axis only ($\boldsymbol{B} = 0$ everywhere else), we can set

$$\boldsymbol{A} = \frac{\Phi}{2\pi}\nabla\theta = \frac{\Phi}{2\pi r}\hat{\boldsymbol{\theta}}$$

where

$$\nabla\theta = \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\theta}{\partial\theta} = \frac{1}{r}\hat{\boldsymbol{\theta}}$$

Hence

$$\boldsymbol{B} = \frac{\Phi}{2\pi}\nabla\times\nabla\theta = 0 \qquad \text{(except for } r = 0\text{)}$$

& $$\Phi = \oint_C d\boldsymbol{r}\cdot\boldsymbol{A}$$

$$= r\int_0^{2\pi} d\theta\, A_\theta \qquad (\, C = \text{circle of radius } r\,)$$

$$= 2\pi r\frac{\Phi}{2\pi r} = \Phi$$

which is also the flux through any plane that crosses the z-axis

$$(\nabla - \alpha\nabla\theta)^2\Psi = \nabla^2\Psi - \alpha\nabla\cdot(\Psi\nabla\theta) - \alpha\nabla\theta\cdot\nabla\Psi + \alpha^2(\nabla\theta)^2\Psi$$

$$= \nabla^2\Psi - 2\alpha\nabla\theta\cdot\nabla\Psi - \alpha\Psi\nabla^2\theta + \alpha^2(\nabla\theta)^2\Psi$$

$$\theta = \tan^{-1}\frac{y}{x}$$

→ $$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = -\frac{y}{r^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

→ $$\frac{\partial^2\theta}{\partial x^2} = -(-2)\frac{y}{r^3}\cdot\frac{x}{r} = 2\frac{xy}{r^4}$$

$$\frac{\partial^2\theta}{\partial y^2} = (-2)\frac{x}{r^3}\cdot\frac{y}{r} = -2\frac{xy}{r^4}$$

∴ $$\nabla^2\theta = 0$$

Also $$\nabla\theta\cdot\nabla\Psi = \frac{1}{r}\hat{\boldsymbol{\theta}}\cdot\nabla\Psi = \frac{1}{r^2}\frac{\partial\Psi}{\partial\theta}$$

Hence

$$(\nabla - \alpha \nabla \theta)^2 \Psi = \nabla^2 \Psi - 2 \frac{\alpha}{r^2} \frac{\partial \Psi}{\partial \theta} + \frac{\alpha^2}{r^2} \Psi$$

With $\alpha = i \dfrac{q \Phi}{2 \pi \hbar c}$, we have

$$H \Psi = -\frac{\hbar^2}{2M} \left(\nabla^2 \Psi - 2 \frac{\alpha}{r^2} \frac{\partial \Psi}{\partial \theta} + \frac{\alpha^2}{r^2} \Psi \right) + V(r) \Psi$$

If we confine the particle to move in a circle of radius R & centered at the axis, then $\Psi = \Psi(\theta)$ so that

$$\nabla^2 \Psi = \frac{1}{R^2} \frac{d^2 \Psi}{d \theta^2}$$

& $$H \Psi = -\frac{\hbar^2}{2 M R^2} \left(\frac{d^2 \Psi}{d \theta^2} - 2 \alpha \frac{d \Psi}{d \theta} + \alpha^2 \Psi \right) + V(R) \Psi$$
$$= E \Psi$$

Ansatz: $\Psi = e^{i n \theta}$, $n = 0, \pm 1, \pm 2, \ldots$ for single-valuedness

→ $$E = \frac{\hbar^2}{2 M R^2} \left(n^2 + 2 i n \alpha - \alpha^2 \right) + V(R)$$
$$= \frac{\hbar^2}{2 M R^2} (n + i \alpha)^2 + V(R)$$
$$= \frac{\hbar^2}{2 M R^2} \left(n - \frac{q \Phi}{2 \pi \hbar c} \right)^2 + V(R)$$

The minimum (ground state) energy is

$$E = V(R)$$

for which

$$n - \frac{q \Phi}{2 \pi \hbar c} = 0$$

i.e., $$\Phi = \frac{2 \pi \hbar c}{q} n = \Phi_L n \qquad \left(\Phi_L = \frac{2 \pi \hbar c}{| q |} \right)$$

Note that q, n, Φ are all signed.

Also, since $n = 0, \pm 1, \pm 2, \ldots$, an $-n$ can be replaced by n.