6.1. Dirac Equation

Ref: M.Kaku, "Quantum Field Theory", Oxford Univ Press (1993)

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

$$p_0 = p^0 \quad \boldsymbol{p} = p^i = -p_i$$

$$p_\mu p^\mu = p_0 p^0 + p_i p^i = \left(\frac{E}{c}\right)^2 - \boldsymbol{p}^2 = (m c)^2$$

→ $$H = c\sqrt{\boldsymbol{p}^2 + (m c)^2}$$

Hamiltonian

Set $H = \boldsymbol{\alpha}\cdot\boldsymbol{p}\, c + \beta m c^2 = -\alpha^i p_i c + \beta m c^2$

$$H^+ = H \quad \rightarrow \quad \alpha^{i\,+} = \alpha^i \qquad \beta^+ = \beta$$

$$p_\mu \sim i\hbar\partial_\mu = i\hbar\left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x^i}\right) = i\hbar\left(\frac{\partial}{c\,\partial t}, \nabla\right)$$

→ $$i\hbar\frac{\partial}{\partial t} = \frac{\hbar c}{i}\boldsymbol{\alpha}\cdot\nabla + \beta m c^2 = \frac{\hbar c}{i}\alpha^i\frac{\partial}{\partial x^i} + \beta m c^2$$

$$H^2 = (\boldsymbol{p}\, c)^2 + (m c^2)^2 = -p_i p^i c^2 + (m c^2)^2$$

$$= \boldsymbol{\alpha}\cdot\boldsymbol{p}\,\boldsymbol{\alpha}\cdot\boldsymbol{p}\, c^2 + (\boldsymbol{\alpha}\cdot\boldsymbol{p}\,\beta + \beta\,\boldsymbol{\alpha}\cdot\boldsymbol{p})\, m c^3 + \beta^2 (m c^2)^2$$

→ $$\boldsymbol{\alpha}\cdot\boldsymbol{p}\,\boldsymbol{\alpha}\cdot\boldsymbol{p} = \boldsymbol{p}^2$$

$$\boldsymbol{\alpha}\cdot\boldsymbol{p}\,\beta + \beta\,\boldsymbol{\alpha}\cdot\boldsymbol{p} = 0$$

$$\beta^2 = 1$$

In component form:

$$\alpha^i p_i \alpha^j p_j = -p^i p_i$$

$$= \frac{1}{2}(\alpha^i \alpha^j + \alpha^j \alpha^i) p_i p_j = p_i p_i$$

→ $$\alpha^i \alpha^j + \alpha^j \alpha^i = \{\alpha^i, \alpha^j\} = 2\delta^{ij}$$

where the anti-commutator is defined by

$$\{A, B\} = [A, B]_+ = A B + B A$$

$$(\alpha^i \beta + \beta\alpha^i) p_i = 0 \quad \rightarrow \quad \alpha^i\beta + \beta\alpha^i = \{\alpha^i, \beta\} = 0$$

Dirac Eq.

$$\beta^2 = 1 \quad \rightarrow \quad i\hbar\beta\frac{\partial}{\partial t} = \frac{\hbar c}{i}\beta\,\boldsymbol{\alpha}\cdot\nabla + m c^2$$

or $$i\hbar\left(\beta\frac{\partial}{c\,\partial t} + \beta\,\boldsymbol{\alpha}\cdot\nabla\right) - m c = 0$$

$$i\hbar\gamma^\mu\partial_\mu - m c = 0 = \gamma^\mu p_\mu - m c$$

where

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha})$$

$$\gamma^{\mu\,+} = (\beta, \boldsymbol{\alpha}\beta) = \beta(\beta, \beta\boldsymbol{\alpha})\beta = \gamma^0\gamma^\mu\gamma^0$$

Dirac Equation:

$$\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi = 0 = \left(\gamma^\mu p_\mu - mc\right)\psi$$

Since $p^\mu p_\mu = (mc)^2$, this is equivalent to writing

$$\sqrt{p^\mu p_\mu} = \gamma^\mu p_\mu = +mc$$

$$\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2\beta^2 = 2$$

$$\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha^i\} = \beta^2\alpha^i + \beta\alpha^i\beta = \beta\{\beta, \alpha^i\} = 0$$

$$\{\gamma^i, \gamma^j\} = \{\beta\alpha^i, \beta\alpha^j\} = \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i$$

$$= -\beta^2\left(\alpha^i\alpha^j + \alpha^j\alpha^i\right)$$

$$= -\{\alpha^i, \alpha^j\} = -2\delta^{ij}$$

$\therefore \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \rightarrow \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$

i.e., $\{\gamma^\mu\}$ forms a Clifford algebra so that ψ is a spinor (spin $\frac{1}{2}$ representation of the Lorentz group).

Note: $\{\gamma^\mu\}$ is just the set of linear combination coefficients for p_μ that gives $\sqrt{p^\mu p_\mu}$. Therefore, they take the same values in all Lorentz frames, i.e., γ^μ is not a 4-vector.
However, as shown later $\overline{\psi}\gamma^\mu\psi$ is a 4-vector.
(See J.J.Sakurai, “Advanced Quantum Mechanics”, §3.4)

Lorentz transformation

Under a Lorentz transformation,

$$x^\mu \rightarrow x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu \qquad \partial_{\mu'} = \Lambda^\nu_{\mu'}\partial_\nu$$

$$\gamma^\mu \rightarrow \gamma^{\mu'} = \gamma^\mu$$

$$\psi \rightarrow \psi' \quad \text{with} \quad \psi'(x') = S(\Lambda)\,\psi(x)$$

where $S(\Lambda)$ is some representation of the Lorentz group.

Since $\Lambda^\nu_{\mu'}$ is the inverse of $\Lambda^{\mu'}_\nu$, some author (e.g., Kaku) prefers to dispense with the primed notation & write

$$\Lambda^\mu_\nu \equiv \Lambda^{\mu'}_\nu \quad \& \quad (\Lambda^{-1})^\mu_\nu \equiv \Lambda^\mu_{\nu'}$$

Thus, $\quad (\Lambda^{-1})^\nu_\mu \partial_\nu = \partial_{\mu'}$

$$\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi(x) = 0$$

$\rightarrow \quad \left(i\hbar\gamma^{\mu'}\partial_{\mu'} - mc\right)\psi'(x') = 0$

$$= \left[i\hbar\gamma^{\mu'}\Lambda^\nu_{\mu'}\partial_\nu - mc\right]S(\Lambda)\,\psi(x)$$

$\rightarrow \quad S^{-1}\left(i\hbar\gamma^{\mu'}\Lambda^\nu_{\mu'}\partial_\nu - mc\right)S\psi = 0$

i.e., $\quad \left(i\hbar S^{-1}\gamma^{\mu'}\Lambda^\nu_{\mu'}S\partial_\nu - mc\right)\psi = 0$

Dirac eq. is covariant under the Lorentz transformation (c.f. Kaku, §3.5).

$\rightarrow \quad S^{-1}\gamma^{\mu'}\Lambda^\nu_{\mu'}S = \gamma^\nu$

or $\quad S\gamma^\nu S^{-1} = \gamma^{\mu'}\Lambda^\nu_{\mu'}$

$$= (\Lambda^{-1})^\nu_\mu\gamma^\mu$$

i.e., $\quad S^{-1}\gamma^\mu S = \Lambda^\mu_\nu\gamma^\nu$

Generator of the Lorentz group is

$$M_{\mu\nu} = L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}$$

where $L_{\mu\nu}$ deals with the space-time part and

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \text{where} \qquad \gamma_\mu = \eta_{\mu\nu}\, \gamma^\nu$$

Hence,

$$S(\Lambda) = \exp\left(-\frac{i}{4}\, \sigma_{\mu\nu}\, \omega^{\mu\nu}\right)$$

where $\omega^{\mu\nu}$ are the antisymmetric parameters of Λ.

Using $\qquad \gamma^{\mu +} = \gamma^0\, \gamma^\mu\, \gamma^0$ & $\qquad \gamma^0\, \gamma^0 = 1 \qquad$ (from § Dirac Eq.)

we have $\qquad \sigma^+_{\mu\nu} = -\frac{i}{2}[\gamma^+_\nu, \gamma^+_\mu] = \frac{i}{2}[\gamma^0\, \gamma_\mu\, \gamma^0, \gamma^0\, \gamma_\nu\, \gamma^0] = \gamma^0\, \sigma_{\mu\nu}\, \gamma^0$

so that $S^+ = \exp\left(\frac{i}{4}\, \gamma^0\, \sigma_{\mu\nu}\, \gamma^0\, \omega^{\mu\nu}\right)$

$$= \gamma^0 \exp\left(\frac{i}{4}\, \sigma_{\mu\nu}\, \omega^{\mu\nu}\right) \gamma^0 \qquad \text{(see below)}$$

$$= \gamma^0\, S^{-1}\, \gamma^0$$

i.e., S is not unitary &

$$S^{-1} = \gamma^0\, S^+\, \gamma^0$$

Note : We can write

$$e^{\gamma^0\, \sigma\, \gamma^0} = \gamma^0\, e^\sigma\, \gamma^0$$

because

$$\gamma^0\, \gamma^0 = 1 \qquad \rightarrow \qquad (\gamma^0\, \sigma\, \gamma^0)^n = \gamma^0\, \sigma^n\, \gamma^0$$

$$\therefore \qquad e^{\gamma^0\, \sigma\, \gamma^0} = 1 + \gamma^0\, \sigma\, \gamma^0 + \frac{1}{2}(\gamma^0\, \sigma\, \gamma^0)^2 + ...$$

$$= \gamma^0\left(1 + \sigma + \frac{1}{2}\sigma^2 + ...\right)\gamma^0 = \gamma^0\, e^\sigma\, \gamma^0$$

Conjugate Field $\overline{\psi}$

$$(i\,\hbar\,\gamma^\mu\,\partial_\mu - m\,c)\,\psi = 0$$

$$\rightarrow \qquad \psi^+\left(i\,\hbar\,\gamma^{\mu +}\,\overleftarrow{\partial}_\mu + m\,c\right) = 0 \qquad (\,f\,\overleftarrow{\partial} \equiv \partial f\,)$$

The appearance of $\gamma^{\mu +}$ makes the pair of eqs unsymmetrical.

More importantly, under a Lorentz transformation Λ,

$$\psi'(x') = S(\Lambda)\, \psi(x)$$

$$\rightarrow \qquad \psi^{+}{}'(x') = \psi^+(x)\, S^+(\Lambda)$$

so that $\psi^{+}{}'(x')\,\psi'(x)' = \psi^+(x)\, S^+\, S\, \psi(x) \neq \psi^+(x)\, \psi(x)$

i.e., $\psi^+\,\psi$ is not a Lorentz scalar.

In order to construct tensor quantities, we introduce the conjugate field

$$\overline{\psi} \equiv \psi^+\, \gamma^0 \qquad \rightarrow \qquad \psi^+ = \overline{\psi}\, \gamma^0$$

Under a Lorentz transformation Λ,

$$\overline{\psi}'(x') = \psi^{+}{}'(x')\,\gamma^0 = \psi^+(x)\, S^+\, \gamma^0 = \overline{\psi}(x)\, \gamma^0\, S^+\, \gamma^0 = \overline{\psi}(x)\, S^{-1}$$

$$\rightarrow \qquad \overline{\psi}'(x')\, \psi'(x') = \overline{\psi}(x)\, S^{-1}\, S\, \psi(x) = \overline{\psi}(x)\, \psi(x)$$

is a Lorentz scalar.

The conjugate eq.

$$\psi^+\left(i\,\hbar\,\gamma^{\mu +}\,\overleftarrow{\partial}_\mu + m\,c\right) = 0$$

can also be rewrite as

$$\overline{\psi}\,\gamma^0\left(i\hbar\,\gamma^{\mu+}\overleftarrow{\partial}_\mu + m\,c\right) = 0$$

$$\rightarrow \qquad \overline{\psi}\left(i\hbar\,\gamma^0\,\gamma^{\mu+}\,\gamma^0\,\overleftarrow{\partial}_\mu + m\,c\right) = 0$$

$$\overline{\psi}\left(i\hbar\,\gamma^{\mu}\,\overleftarrow{\partial}_\mu + m\,c\right) = 0$$

which is preferred form.

γ_5

It's useful to introduce one more γ matrix

$$\gamma_5 \equiv \gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \;\; = -i\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3$$

(Index 5 can't be raised or lowered using $\eta_{\mu\nu}$ since $\mu,\ \nu = 0,\ 1,\ 2,\ 3$)

$$\gamma^\mu\,\gamma^\nu = -\gamma^\nu\,\gamma^\mu \qquad \forall \qquad \mu \neq \nu$$

$$\rightarrow \qquad \gamma^\mu\,\gamma_5 = -\gamma_5\,\gamma^\mu \qquad \text{(changes sign 3 times)}$$

$$\& \qquad \gamma_5 = -\frac{i}{4!}\,\varepsilon_{\mu\nu\sigma\tau}\,\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\tau$$

where $\varepsilon_{\mu\nu\sigma\tau} = -\varepsilon^{\mu\nu\sigma\tau}$ & $\varepsilon^{0123} = 1$.

(Using $\varepsilon_{\mu\nu\sigma\tau}$ instead of $\varepsilon^{\mu\nu\sigma\tau}$ honors the Einstein summation rule but introduces a minus sign).

$$\{\gamma^\mu,\ \gamma^\nu\} = 2\,\eta^{\mu\nu}$$

$$\rightarrow \qquad \gamma_5^2 = -\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$$

$$= \gamma^1\,\gamma^2\,\gamma^3\,\gamma^1\,\gamma^2\,\gamma^3 \;\; = -\gamma^2\,\gamma^3\,\gamma^2\,\gamma^3 \;\; = -\gamma^3\,\gamma^3 = 1$$

$$\gamma^{\mu+} = \gamma^0\,\gamma^\mu\,\gamma^0 \qquad \gamma^0\,\gamma^0 = 1$$

$$\rightarrow \qquad \gamma_5^+ = -i\,\gamma^{3+}\,\gamma^{2+}\,\gamma^{1+}\,\gamma^{0+} \;\; = -i\,\gamma^0\,\gamma^3\,\gamma^2\,\gamma^1\,\gamma^0\,\gamma^0 \;\; = i\,\gamma^3\,\gamma^2\,\gamma^1\,\gamma^0 \;\; = \gamma_5$$

Tensors

Similarly, one can construct other types of tensors as follows

type	form	number
scalar	$\overline{\psi}(x)\,\psi(x)$	1
vector	$\overline{\psi}(x)\,\gamma^\mu\,\psi(x)$	4
tensor	$\overline{\psi}(x)\,\sigma^{\mu\nu}\,\psi(x)$	6
pseudovector	$\overline{\psi}(x)\,\gamma_5\,\gamma^\mu\,\psi(x)$	4
pseudoscalar	$\overline{\psi}(x)\,\gamma_5\,\psi(x)$	1

For example, using $S^{-1}\,\gamma^\mu\,S = \Lambda^\mu_\nu\,\gamma^\nu$, we have

$$\overline{\psi}'(x')\,\gamma^\mu\,\psi'(x') = \overline{\psi}(x)\,S^{-1}\,\gamma^\mu\,S\psi(x) \;\; = \Lambda^\mu_\nu\,\overline{\psi}(x)\,\gamma^\nu\,\psi(x)$$

so $\overline{\psi}\,\gamma^\mu\,\psi$ indeed transforms as a vector.

γ_5 is a pseudoscalar because

$$\gamma^\mu\,\gamma^\nu = -\gamma^\nu\,\gamma^\mu \qquad \forall \qquad \mu \neq \nu$$

$$\rightarrow \qquad \gamma_5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = -\frac{i}{4!}\,\varepsilon_{\mu\nu\sigma\tau}\,\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\tau$$

$$\therefore \qquad S^{-1}\,\gamma_5\,S = -\frac{i}{4!}\,\varepsilon_{\mu\nu\sigma\tau}\,S\,\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\tau\,S^{-1}$$

$$= -\frac{i}{4!}\,\varepsilon_{\mu\nu\sigma\tau}\,\Lambda^{\mu}_{\mu'}\,\Lambda^{\nu}_{\nu'}\,\Lambda^{\sigma}_{\sigma'}\,\Lambda^{\tau}_{\tau'}\,\gamma^{\mu'}\,\gamma^{\nu'}\,\gamma^{\sigma'}\,\gamma^{\tau'}$$
$$= -\frac{i}{4!}\,(\det\Lambda)\,\gamma^{\mu'}\,\gamma^{\nu'}\,\gamma^{\sigma'}\,\gamma^{\tau'}$$
$$= (\det\Lambda)\,\gamma_5$$

Spin

Spin is given by

$$s_i = \frac{\hbar}{4}\,\epsilon_{ijk}\,\sigma_{jk} = i\,\frac{\hbar}{8}\,\epsilon_{ijk}[\gamma_j,\,\gamma_k] = s^i$$

i.e., $$s_1 = \frac{1}{2}\,\hbar\,\sigma_{23} = i\,\frac{\hbar}{4}\,[\gamma_2,\,\gamma_3] = i\,\frac{\hbar}{4}\,(\gamma_2\,\gamma_3 - \gamma_3\,\gamma_2) = i\,\frac{\hbar}{2}\,\gamma_2\,\gamma_3$$

$$[s_1,\,s_2] = -\frac{\hbar^2}{4}\,[\,\gamma_2\,\gamma_3\,,\ \gamma_3\,\gamma_1\,] = -\frac{\hbar^2}{4}\,(\,\gamma_2\,\gamma_3\,\gamma_3\,\gamma_1 - \gamma_3\,\gamma_1\,\gamma_2\,\gamma_3\,)$$
$$= -\frac{\hbar^2}{4}\,(\,-\gamma_2\,\gamma_1 + \gamma_1\,\gamma_2\,) = -\frac{\hbar^2}{2}\,\gamma_1\,\gamma_2 = i\,\hbar\,s_3$$

Similarly, $$[s_i,\,s_j] = i\,\hbar\,\varepsilon_{ijk}\,s_k$$

Standard Representation

4-D (standard) representation of $\{\,\gamma^{\mu}\,\}$:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_0 \qquad\qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\gamma_i$$

where $\{\sigma^i\}$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy

$$[\,\sigma^i,\,\sigma^j\,] = 2\,i\,\varepsilon^{ijk}\,\sigma^k \qquad\qquad \{\,\sigma^i,\,\sigma^j\,\} = 2\,\delta^{ij}\,I$$
$$\sigma^i\,\sigma^j = i\,\varepsilon^{ijk}\,\sigma^k + \delta^{ij}\,I$$

with I being the 2×2 identity matrix.

∴ $$s_3 = i\,\frac{\hbar}{2}\,\gamma_1\,\gamma_2 = i\,\frac{\hbar}{2}\,\gamma^1\,\gamma^2 = i\,\frac{\hbar}{2}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$
$$= -i\,\frac{\hbar}{2}\begin{pmatrix} \sigma^1\,\sigma^2 & 0 \\ 0 & \sigma^1\,\sigma^2 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

$$\gamma_5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma^2\,\sigma^3 & 0 \\ 0 & -\sigma^2\,\sigma^3 \end{pmatrix}$$
$$= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & -\sigma^1\,\sigma^2\,\sigma^3 \\ \sigma^1\,\sigma^2\,\sigma^3 & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} 0 & -\sigma^1\,\sigma^2\,\sigma^3 \\ -\sigma^1\,\sigma^2\,\sigma^3 & 0 \end{pmatrix}$$

$$\sigma^1\,\sigma^2\,\sigma^3 = i\,\sigma^3\,\sigma^3 = i$$

$$\rightarrow \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$H = \boldsymbol{\alpha}\cdot\mathbf{p}\,c + \beta\, m\, c^2 \qquad \& \qquad \gamma^\mu = (\beta,\ \beta\,\boldsymbol{\alpha}) = (\gamma^0,\ \gamma^0\,\boldsymbol{\alpha})$$

$$\rightarrow \qquad H = c\gamma^0(\boldsymbol{\gamma}\cdot\mathbf{p} + m\,c)$$

$$\boldsymbol{\gamma}\cdot\mathbf{p} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix}$$

$$\rightarrow \qquad H = c\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} m\,c & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & m\,c \end{pmatrix}$$

$$= c\begin{pmatrix} m\,c & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -m\,c \end{pmatrix}$$

Lagrangian

Since $\mathcal{L}$ must be a Lorentz scalar, we write

$$\mathcal{L} = c\,\overline{\psi}\,(i\,\hbar\,\gamma^\mu\,\partial_\mu - m\,c)\,\psi$$

which gives the Dirac & its conjugate eqs trivially.

$$\overline{\psi} = \psi^+\,\gamma^0 \qquad \rightarrow \qquad \overline{\psi}^+ = \gamma^0\,\psi \qquad \text{since } \gamma^{0\,+} = \gamma^0$$

$$\therefore \qquad (\overline{\psi}\,\psi)^+ = \psi^+\,\overline{\psi}^+ = \psi^+\,\gamma^0\,\psi = \overline{\psi}\,\psi$$

i.e., $\overline{\psi}\,\psi$ is hermitian.

Also,

$$(\overline{\psi}\,\gamma^\mu\,\psi)^+ = \psi^+\,\gamma^{\mu\,+}\,\overline{\psi}^+ = \psi^+\,\gamma^0\,\gamma^\mu\,\gamma^0\,\overline{\psi}^+ = \overline{\psi}\,\gamma^\mu\,\psi$$

where $\gamma^0\,\overline{\psi}^+ = \gamma^0\,\gamma^0\,\psi = \psi$ was used.

Hence, $\mathcal{L}$ is also hermitian.

Noether Currents

See 3.5._NoetherCurrents.pdf .

$\mathcal{L}$ is invariant under a global phase transformation

$$\psi(x) \rightarrow e^{-i\,\epsilon}\,\psi(x) \qquad \epsilon = \text{real const}$$

$$\therefore \qquad \psi^+(x) \rightarrow e^{i\,\epsilon}\,\psi^+(x)$$

$$\overline{\psi}(x) \rightarrow e^{i\,\epsilon}\,\overline{\psi}(x)$$

For infinitesimal transformation

$$\psi(x) \rightarrow (1 - i\,\epsilon)\,\psi(x)$$

$$\delta_s\,\psi(x) = -i\,\epsilon\,\psi(x) \equiv \epsilon\,\Delta$$

$$\delta_s\,\overline{\psi}(x) = i\,\epsilon\,\overline{\psi}(x) \equiv \epsilon\,\overline{\Delta}$$

Conserved Noether current is

$$j^\mu = \frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\,\psi}\,\Delta + \frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\,\overline{\psi}}\,\overline{\Delta}$$

$$= -i\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\,\psi}\,\psi + i\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\mu\,\overline{\psi}}\,\overline{\psi}$$

$$= \hbar\,c\,\overline{\psi}\,\gamma^\mu\,\psi$$

Chiral transformation is defined as

$$\psi(x) \rightarrow e^{-i\,\epsilon\,\gamma_5}\,\psi(x) \qquad \epsilon = \text{real const}$$

∴ $\psi^{+}(x) \to \psi^{+}(x)\, e^{i\epsilon\gamma_5}$ $\quad(\gamma_5^{+} = \gamma_5)$

$\overline{\psi}(x) \to \psi^{+}(x)\, e^{-i\epsilon\gamma_5}\gamma_0 = \overline{\psi}(x)\, e^{-i\epsilon\gamma_5}$ $\quad(\gamma_5\gamma_0 = -\gamma_0\gamma_5)$

$$\overline{\psi}(x)\,\psi(x) \to \overline{\psi}(x)\, e^{-2i\epsilon\gamma_5}\,\psi(x)$$

$$\overline{\psi}(x)\,\gamma^{\mu}\,\psi(x) \to \overline{\psi}(x)\, e^{-i\epsilon\gamma_5}\gamma^{\mu} e^{-i\epsilon\gamma_5}\,\psi(x) = \overline{\psi}(x)\,\gamma^{\mu}\,\psi(x)$$

since

$$\gamma_5\gamma^{\mu} = -\gamma^{\mu}\gamma_5 \quad\to\quad \gamma^{\mu} e^{-i\epsilon\gamma_5} = e^{i\epsilon\gamma_5}\gamma^{\mu}$$

Hence

$$\mathcal{L} = c\,\overline{\psi}\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\right)\psi$$

$$\to\quad \mathcal{L}' = c\,\overline{\psi}\left(i\hbar\gamma^{\mu}\partial_{\mu} - e^{-2i\epsilon\gamma_5}\,mc\right)\psi$$

which means $\mathcal{L}$ is chiral invariant only if $m = 0$.

In which case, conserved Noether current is

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial\,\partial_{\mu}\psi}\Delta + \frac{\partial\mathcal{L}}{\partial\,\partial_{\mu}\overline{\psi}}\overline{\Delta}$$

$$= -i\,\frac{\partial\mathcal{L}}{\partial\,\partial_{\mu}\psi}\gamma_5\,\psi - i\,\frac{\partial\mathcal{L}}{\partial\,\partial_{\mu}\overline{\psi}}\overline{\psi}\,\gamma_5$$

$$= \hbar c\,\overline{\psi}\,\gamma^{\mu}\gamma_5\,\psi$$ (for massless particles only)

Since γ_5 is a pseudo-scalar, this is a pseudo-current.