6.2. Plane Wave Solutions

From §6.1, we have

$$i\hbar\partial_t\psi = H\psi$$

where

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{p}\,c + \beta m c^2$$

$$\gamma^\mu = (\beta,\ \beta\boldsymbol{\alpha}) \quad \& \quad \beta^2 = I$$

$\rightarrow \quad \alpha^i = \beta\gamma^i \quad$ or $\quad \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$

$\therefore \quad H = \gamma^0(-\gamma^i p_i c + m c^2) = \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{p}\,c + m c^2)$

Caution: Although we manipulate the components of γ^μ as if it were a 4-vector, γ^μ is NOT a 4-vector according to its behavior under a Lorentz transformation.

In the standard representation,

$$\beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_0 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\gamma_i$$

we thus have

$$\boldsymbol{\gamma}\cdot\boldsymbol{p} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}$$

$$\boldsymbol{\alpha}\cdot\boldsymbol{p} = \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}$$

$$\rightarrow \quad H = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} c + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m c^2$$

$$= c\begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m c \end{pmatrix}$$

$$= c\begin{pmatrix} m c & -i\hbar\boldsymbol{\sigma}\cdot\nabla \\ -i\hbar\boldsymbol{\sigma}\cdot\nabla & -m c \end{pmatrix}$$

Solutions in Rest Frame : $u_\sigma(0)$, $v_\sigma(0)$

For $m \neq 0$, we can work in the rest frame of the particle so that $\boldsymbol{p} = 0$ and

$$H \rightarrow H_0 = m c^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Thus, the upper & lower 2 components of a 4-D spinor (column vector) denote particle $(E = m c^2)$ & anti-particle $(E = -m c^2)$ states, respectively.

The natural eigenvectors for H_0 are

$$u_\uparrow(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_\uparrow \\ 0 \end{pmatrix} \qquad u_\downarrow(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_\downarrow \\ 0 \end{pmatrix}$$

$$v_\uparrow(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_\uparrow \end{pmatrix} \qquad v_\downarrow(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_\downarrow \end{pmatrix}$$

where u & v denotes particle & anti-particle, respectively.

Subscripts ↑ & ↓ refers to the Pauli spin states.

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Validity of this designation is provided later.

The time dependent versions are

$$\psi_\sigma^{(+)}(t) = \exp\left(-\frac{i}{\hbar} m c^2 t \right) u_\sigma(0) \qquad (\sigma = \uparrow, \downarrow)$$

$$\psi_\sigma^{(-)}(t) = \exp\left(\frac{i}{\hbar} m c^2 t \right) v_\sigma(0)$$

where the superscript $(\pm)$ denotes a state with $E \gtrless 0$.

General Solutions : $u_\sigma(\boldsymbol{k}), \bar{u}_\sigma(\boldsymbol{k}), v_\sigma(\boldsymbol{k}), \bar{v}_\sigma(\boldsymbol{k})$

States of momentum $\boldsymbol{p}$ can be obtained by Lorentz transformation

$$p_\mu = (\pm m c, \boldsymbol{0}) \rightarrow \quad \pm (E_{\boldsymbol{p}}/c, -\boldsymbol{p}) = \pm \hbar (\omega_{\boldsymbol{k}}/c, -\boldsymbol{k})$$

$$p \cdot x = \pm m c^2 t \rightarrow \quad \pm \hbar (\omega_{\boldsymbol{k}} t' - \boldsymbol{k} \cdot \boldsymbol{r}')$$

where $\qquad \omega_{\boldsymbol{k}} = c \sqrt{\boldsymbol{k}^2 + \left(\frac{m c}{\hbar} \right)^2} = \frac{E_{\boldsymbol{p}}}{\hbar}$

Note: $p^\mu = (E/c, \boldsymbol{p})$ can be used either as a quantum operator or its eigenvalue.

To avoid confusion, we'll use $k^\mu = (\omega_{\boldsymbol{k}}/c, \boldsymbol{k})$ to denote the value of $p^\mu / \hbar$.

Hence,

$$\psi_\sigma^{(+)}(t) = \exp(-i k \cdot x) u_\sigma(\boldsymbol{k})$$

$$\psi_\sigma^{(-)}(t) = \exp(i k \cdot x) v_\sigma(\boldsymbol{k})$$

where $\qquad k \cdot x = k_\mu x^\mu = \omega_{\boldsymbol{k}} t - \boldsymbol{k} \cdot \boldsymbol{r}$.

$$\therefore \qquad i \hbar \gamma^\mu \partial_\mu \psi_\sigma^{(\pm)}(t) = \pm \hbar \gamma^\mu k_\mu \psi_\sigma^{(\pm)}(t)$$

Dirac eq.

$$(i \hbar \gamma^\mu \partial_\mu - m c) \psi = 0$$

becomes

$$(\pm \gamma^\mu p_\mu - m c) \psi_\sigma^{(\pm)}(t) = 0$$

$$\therefore \qquad (\gamma^\mu k_\mu - m^*) u_\sigma(\boldsymbol{k}) = 0 \quad \& \quad (\gamma^\mu k_\mu + m^*) v_\sigma(\boldsymbol{k}) = 0$$

where

$$m^* = \frac{m c}{\hbar}$$

Taking the adjoint, we have

$$u_\sigma^+(\boldsymbol{k}) (\gamma^{\mu +} k_\mu - m^*) = 0 \qquad v_\sigma^+(\boldsymbol{k}) (\gamma^{\mu +} k_\mu + m^*) = 0$$

Using

$$\bar{u} = u^+ \gamma^0 \qquad \bar{v} = v^+ \gamma^0$$

$$\gamma^0 \gamma^0 = 1 \qquad \gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0$$

$$\rightarrow \qquad u^+ = \bar{u} \gamma^0 \qquad v^+ = \bar{v} \gamma^0$$

we have

$$\bar{u}_\sigma(\boldsymbol{k}) (\gamma^\mu k_\mu - m^*) \gamma^0 = 0 \qquad \bar{v}_\sigma(\boldsymbol{k}) (\gamma^\mu k_\mu + m^*) \gamma^0 = 0$$

or simply

$$\bar{u}_\sigma(\boldsymbol{k})\left(\gamma^\mu k_\mu - m^*\right) = 0 \qquad \bar{v}_\sigma(\boldsymbol{k})\left(\gamma^\mu k_\mu + m^*\right) = 0$$

We can use either $\{u^+, v^+\}$ or $\{\bar{u}, \bar{v}\}$ for normalization.
The natural choice is the latter since the result will be Lorentz covariant.

Working first in the rest frame, we have

$$\bar{u}_\sigma(0) = \left(\chi_\sigma^T, 0\right)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \left(\chi_\sigma^T, 0\right)$$

$$\bar{v}_\sigma(0) = \left(0, \chi_\sigma^T\right)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \left(0, -\chi_\sigma^T\right)$$

$$\rightarrow \qquad \bar{u}_\sigma(0)\, u_{\sigma'}(0) = \left(\chi_\sigma^T, 0\right)\begin{pmatrix} \chi_{\sigma'} \\ 0 \end{pmatrix} = \chi_\sigma^T \chi_{\sigma'} = \delta_{\sigma\sigma'}$$

$$\bar{v}_\sigma(0)\, v_{\sigma'}(0) = \left(0, -\chi_\sigma^T\right)\begin{pmatrix} 0 \\ \chi_{\sigma'} \end{pmatrix} = -\chi_\sigma^T \chi_{\sigma'} = -\delta_{\sigma\sigma'}$$

$$\bar{u}_\sigma(0)\, v_{\sigma'}(0) = \left(\chi_\sigma^T, 0\right)\begin{pmatrix} 0 \\ \chi_{\sigma'} \end{pmatrix} = 0$$

$$\bar{v}_\sigma(0)\, u_{\sigma'}(0) = \left(0, -\chi_\sigma^T\right)\begin{pmatrix} \chi_{\sigma'} \\ 0 \end{pmatrix} = 0$$

Hence, we set

$$\bar{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'} \qquad \bar{v}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = -\delta_{\sigma\sigma'}$$

$$\bar{u}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad \bar{v}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'}$$

Since

$$\sum_\sigma \chi_\sigma \chi_\sigma^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the completeness condition is obviously

$$\sum_\sigma \left[u_\sigma(\boldsymbol{k})\, \bar{u}_\sigma(\boldsymbol{k}) - v_\sigma(\boldsymbol{k})\, \bar{v}_\sigma(\boldsymbol{k}) \right] = I$$

The solution to

$$\left(\gamma^\mu k_\mu - m^*\right) u_\sigma(\boldsymbol{k}) = 0 \quad \& \quad \left(\gamma^\mu k_\mu + m^*\right) v_\sigma(\boldsymbol{k}) = 0$$

is easily obtained by using (see 6.1._DiracEquation.pdf)

$$\left(\gamma^\mu k_\mu - m^*\right)\left(\gamma^\mu k_\mu + m^*\right) = \left(\gamma^\mu k_\mu + m^*\right)\left(\gamma^\mu k_\mu - m^*\right) = 0$$

so that

$$u_\sigma(\boldsymbol{k}) = a\left(\gamma^\mu k_\mu + m^*\right) u_\sigma(0)$$

$$v_\sigma(\boldsymbol{k}) = b\left(\gamma^\mu k_\mu - m^*\right) v_\sigma(0)$$

where a, b are constants.

Using

$$\gamma^0 k_0 = \begin{pmatrix} k_0 & 0 \\ 0 & -k_0 \end{pmatrix} = \frac{1}{c}\begin{pmatrix} \omega_{\boldsymbol{k}} & 0 \\ 0 & -\omega_{\boldsymbol{k}} \end{pmatrix} \qquad \boldsymbol{\gamma}\cdot\boldsymbol{k} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix}$$

we have

$$\gamma^\mu k_\mu + m^* = \begin{pmatrix} k_0 + m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k_0 + m^* \end{pmatrix}$$

$$u_\sigma(\boldsymbol{k}) = a\begin{pmatrix} k_0 + m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k_0 + m^* \end{pmatrix}\begin{pmatrix} \chi_\sigma \\ 0 \end{pmatrix}$$

$$= a\begin{pmatrix} (k_0 + m^*)\, \chi_\sigma \\ \boldsymbol{\sigma}\cdot\boldsymbol{k}\; \chi_\sigma \end{pmatrix}$$

$$H(\boldsymbol{p}) = c\begin{pmatrix} m\,c & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m\,c \end{pmatrix} = \hbar\,c\begin{pmatrix} m^* & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -m^* \end{pmatrix}$$

$$\rightarrow \qquad H(\boldsymbol{k})\,u_\sigma(\boldsymbol{k}) = a\,\hbar\,c\begin{pmatrix} m^* & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -m^* \end{pmatrix}\begin{pmatrix} (k_0 + m^*)\,\chi_\sigma \\ \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\chi_\sigma \end{pmatrix}$$

$$= a\,\hbar\,c\begin{pmatrix} \left[\,m^*\,(k_0+m^*) + (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2\,\right]\chi_\sigma \\ k_0\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\chi_\sigma \end{pmatrix}$$

Using

$$(\,\boldsymbol{a}\cdot\boldsymbol{\sigma})\,(\boldsymbol{b}\cdot\boldsymbol{\sigma}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\,(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\sigma}$$

we have

$$(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 = \boldsymbol{k}^2$$

so that

$$m^*\,(k_0+m^*) + (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 = m^*\,(k_0+m^*) + \boldsymbol{k}^2$$

$$= m^*\,k_0 + k_0^2 \qquad \text{since} \quad k_0^2 = \boldsymbol{k}^2 + m^{*\,2}$$

$$= k_0\,(k_0 + m^*)$$

$$\therefore \qquad H(\boldsymbol{k})\,u_\sigma(\boldsymbol{k}) = a\,\hbar\,c\begin{pmatrix} k_0\,(k_0+m^*)\,\chi_\sigma \\ k_0\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\chi_\sigma \end{pmatrix}$$

$$= \hbar\,k_0\,c\,u_\sigma(\boldsymbol{k})$$

$$= \hbar\,\omega_{\boldsymbol{k}}\,u_\sigma(\boldsymbol{k}) \qquad \text{as promised}$$

$$\overline{u}_\sigma(\boldsymbol{k}) = u_\sigma^+(\boldsymbol{k})\,\gamma^0$$

$$= a^*\left(\,(k_0+m^*)\,\chi_\sigma^T\,,\;\chi_\sigma^T\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\right)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$= a^*\left(\,(k_0+m^*)\,\chi_\sigma^T\,,\,-\chi_\sigma^T\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\right)$$

$$\therefore \qquad \overline{u}_\sigma(\boldsymbol{k})\,u_{\sigma'}(\boldsymbol{k}) = a^*\,a\,\chi_\sigma^T\left[\,(k_0+m^*)^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2\,\right]\chi_{\sigma'}$$

$$= a^*\,a\left[\,(k_0+m^*)^2 - \boldsymbol{k}^2\right]\chi_\sigma^T\,\chi_{\sigma'}$$

$$= a^*\,a\left(2\,k_0\,m^* + 2\,m^{*\,2}\right)\delta_{\sigma\sigma'}$$

$$\overset{\text{set}}{=} \delta_{\sigma\sigma'} \qquad \rightarrow \qquad a = \frac{1}{\sqrt{2\,m^*\,(\,k_0+m^*)}} \qquad (\text{ Assuming } a \text{ real })$$

$$\therefore \qquad u_\sigma(\boldsymbol{k}) = \frac{1}{\sqrt{2\,m^*\,(\,k_0+m^*)}}\begin{pmatrix} (k_0+m^*)\,\chi_\sigma \\ \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\chi_\sigma \end{pmatrix}$$

$$= A_{\boldsymbol{k}}\begin{pmatrix} \chi_\sigma \\ B_{\boldsymbol{k}}\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\;\chi_\sigma \end{pmatrix}$$

where

$$A_{\boldsymbol{k}} = \sqrt{\frac{k_0+m^*}{2\,m^*}} \qquad B_{\boldsymbol{k}} = \frac{1}{k_0+m^*}$$

$$\rightarrow \qquad 1 - \boldsymbol{k}^2\,B_{\boldsymbol{k}}^2 = \frac{(k_0+m^*)^2 - \boldsymbol{k}^2}{(k_0+m^*)^2} = \frac{2\,m^*}{k_0+m^*} = \frac{1}{A_{\boldsymbol{k}}^2}$$

$$\therefore \qquad A_k^2\left(1 - \boldsymbol{k}^2\,B_{\boldsymbol{k}}^2\right) = 1$$

$$\overline{u}_\sigma(\boldsymbol{k}) = \frac{1}{\sqrt{2\,m^*\,(\,k_0+m^*)}}\left(\,(k_0+m^*)\,\chi_\sigma^T\,,\,-\chi_\sigma^T\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\right)$$

$$= A_{\boldsymbol{k}} \left(\chi_\sigma^T , -B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma} \cdot \boldsymbol{k} \right)$$

so that

$$\overline{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(1 - B_k^2 \boldsymbol{k}^2 \right) \chi_\sigma^T \chi_{\sigma'} = \delta_{\sigma\sigma'} \qquad \text{as expected.}$$

Similarly

$$\gamma^\mu k_\mu - m^* = \begin{pmatrix} k_0 - m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k_0 - m^* \end{pmatrix}$$

$$v_\sigma(\boldsymbol{k}) = b \begin{pmatrix} k_0 - m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & -k_0 - m^* \end{pmatrix} \begin{pmatrix} 0 \\ \chi_\sigma \end{pmatrix}$$

$$= -b \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\chi_\sigma \\ (k_0 + m^*)\, \chi_\sigma \end{pmatrix}$$

Hence

$$H(\boldsymbol{p}) = c \begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m c \end{pmatrix} = \hbar c \begin{pmatrix} m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & -m^* \end{pmatrix}$$

$$\rightarrow \qquad H(-\boldsymbol{k})\, v_\sigma(\boldsymbol{k}) = -b\, \hbar\, c \begin{pmatrix} m^* & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & -m^* \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\chi_\sigma \\ (k_0 + m^*)\, \chi_\sigma \end{pmatrix}$$

$$= -b\, \hbar\, c \begin{pmatrix} -k_0\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_\sigma \\ -\left[m^* (k_0 + m^*) + (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 \right] \chi_\sigma \end{pmatrix}$$

$$= -\hbar\, \omega_{\boldsymbol{k}}\, u_\sigma(\boldsymbol{k}) \qquad \text{as expected.}$$

Note the use of $H(-\boldsymbol{k})$ instead of $H(\boldsymbol{k})$ since we're implicitly dealing with $\psi_\sigma^{(-)}(t) \propto \exp(-i\, \boldsymbol{k}\cdot\boldsymbol{r})$ instead of $\psi_\sigma^{(+)}(t) \propto \exp(i\, \boldsymbol{k}\cdot\boldsymbol{r})$.

$$\rightarrow \qquad \overline{v}_\sigma(\boldsymbol{k}) = v_\sigma^+(\boldsymbol{k})\, \gamma^0$$

$$= -b^* \left(\chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k},\ (k_0 + m^*)\, \chi_\sigma^T \right) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$= -b^* \left(\chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k},\ -(k_0 + m^*)\, \chi_\sigma^T \right)$$

$$\therefore \qquad \overline{v}_\sigma(\boldsymbol{p})\, v_{\sigma'}(\boldsymbol{p}) = b^*\, b \left[\boldsymbol{k}^2 - (k_0 + m^*)^2 \right] \delta_{\sigma\sigma'}$$

$$= -2\, m^* \left(k_0 + m^* \right) b^*\, b\, \delta_{\sigma\sigma'}$$

$$\overset{\text{set}}{=} -\delta_{\sigma\sigma'}$$

which can be satisfied by setting

$$-b = a = \frac{1}{\sqrt{2\, m \left(\hbar\, \omega_{\boldsymbol{p}} + m c^2 \right)}}$$

so that

$$v_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} B_{\boldsymbol{k}}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$\overline{v}_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(B_{\boldsymbol{k}}\, \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k} ,\ -\chi_\sigma^T \right)$$

$$\rightarrow \qquad \overline{v}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(B_k^2\, \boldsymbol{k}^2 - 1 \right) \chi_\sigma^T \chi_{\sigma'} = -\delta_{\sigma\sigma'}$$

Also

$$\overline{u}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(B_{\boldsymbol{k}}\, \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'} - B_{\boldsymbol{k}}\, \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'} \right) = 0$$

$$\overline{v}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(B_{\boldsymbol{k}}\, \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'} - B_{\boldsymbol{k}}\, \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'} \right) = 0$$

as it should be.

Explicit Expressions for $u_\sigma(\boldsymbol{k})$, $v_\sigma(\boldsymbol{k})$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\to \qquad \boldsymbol{\sigma} \cdot \boldsymbol{k} = \begin{pmatrix} k_z & k_x - i\,k_y \\ k_x + i\,k_y & -k_z \end{pmatrix} = \begin{pmatrix} k_z & k_- \\ k_+ & -k_z \end{pmatrix}$$

where $\qquad k_\pm = k_x \pm i\,k_y$

$$\to \qquad \boldsymbol{\sigma} \cdot \boldsymbol{k}\ \chi_\uparrow = \begin{pmatrix} k_z & k_- \\ k_+ & -k_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_z \\ k_+ \end{pmatrix}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{k}\ \chi_\downarrow = \begin{pmatrix} k_z & k_- \\ k_+ & -k_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k_- \\ -k_z \end{pmatrix}$$

$$u_\sigma(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} \chi_\sigma \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{k}}{k_0 + m^*}\ \chi_\sigma \end{pmatrix} \qquad v_\sigma(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{k}}{k_0 + m^*}\ \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$\to \qquad u_\uparrow(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} 1 \\ 0 \\ \dfrac{k_z}{k_0 + m^*} \\ \dfrac{k_+}{k_0 + m^*} \end{pmatrix} \qquad u_\downarrow(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} 0 \\ 1 \\ \dfrac{k_-}{k_0 + m^*} \\ \dfrac{-k_z}{k_0 + m^*} \end{pmatrix}$$

$$v_\uparrow(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} \dfrac{k_z}{k_0 + m^*} \\ \dfrac{k_+}{k_0 + m^*} \\ 1 \\ 0 \end{pmatrix} \qquad v_\downarrow(\boldsymbol{k}) = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \begin{pmatrix} \dfrac{k_-}{k_0 + m^*} \\ \dfrac{-k_z}{k_0 + m^*} \\ 0 \\ 1 \end{pmatrix}$$

General Solutions : $u_\sigma(\boldsymbol{k}),\ u_\sigma^+(\boldsymbol{k}),\ v_\sigma(\boldsymbol{k}),\ v_\sigma^+(\boldsymbol{k})$

$$u_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(\chi_\sigma^T ,\ B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma} \cdot \boldsymbol{k} \right) = \overline{u}_\sigma(-\boldsymbol{k})$$

$$v_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma} \cdot \boldsymbol{k} ,\ \chi_\sigma^T \right) = -\overline{v}_\sigma(-\boldsymbol{k})$$

$$\overline{u}_\sigma(\boldsymbol{k}) \left(\gamma^\mu k_\mu - m^* \right) = 0 \qquad \overline{v}_\sigma(\boldsymbol{k}) \left(\gamma^\mu k_\mu + m^* \right) = 0$$

$$\to \qquad u_\sigma^+(-\boldsymbol{k}) \left(\gamma^\mu k_\mu - m^* \right) = 0 \qquad v_\sigma^+(-\boldsymbol{k}) \left(\gamma^\mu k_\mu + m^* \right) = 0$$

$$\overline{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(1 - B_k^2\, \boldsymbol{k}^2 \right) \chi_\sigma^T\, \chi_{\sigma'} = \delta_{\sigma\sigma'}$$

$$\to \qquad u_\sigma^+(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(1 + B_k^2\, \boldsymbol{k}^2 \right) \chi_\sigma^T\, \chi_{\sigma'}$$

$$1 + B_k^2\, \boldsymbol{k}^2 = \frac{(k_0 + m^*)^2 + \boldsymbol{k}^2}{(k_0 + m^*)^2} = \frac{2\,k_0^2 + 2\,k_0\,m^*}{(k_0 + m^*)^2} = \frac{2\,k_0}{k_0 + m^*}$$

$$\to \qquad A_k^2 (1 + \boldsymbol{k}^2\, B_{\boldsymbol{k}}^2) = \frac{k_0}{m^*}$$

$$\therefore \qquad u_\sigma^+(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*}\, \delta_{\sigma\sigma'}$$

Similarly,

$$\bar{v}_\sigma(\boldsymbol{p})\, v_{\sigma'}(\boldsymbol{p}) = A_{\boldsymbol{k}}^2 \left(B_k^2\, \boldsymbol{k}^2 - 1\right) \delta_{\sigma\sigma'} = -\delta_{\sigma\sigma'}$$

$$\rightarrow \qquad v_\sigma^+(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*}\, \delta_{\sigma\sigma'}$$

$$\bar{u}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad \bar{v}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = 0$$

$$\rightarrow \qquad u_\sigma^+(-\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad v_\sigma^+(-\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = 0$$

Summary

$$\left(\gamma^\mu\, k_\mu - m^*\right) u_\sigma(\boldsymbol{k}) = 0 \qquad \left(\gamma^\mu\, k_\mu + m^*\right) v_\sigma(\boldsymbol{k}) = 0$$

$$\bar{u}_\sigma(\boldsymbol{k}) \left(\gamma^\mu\, k_\mu - m^*\right) = 0 \qquad \bar{v}_\sigma(\boldsymbol{k}) \left(\gamma^\mu\, k_\mu + m^*\right) = 0$$

$$u_\sigma^+(\boldsymbol{k}) \left(\gamma^{\mu+}\, k_\mu - m^*\right) = 0 \qquad v_\sigma^+(\boldsymbol{k}) \left(\gamma^{\mu+}\, k_\mu + m^*\right) = 0$$

$$A_{\boldsymbol{k}} = \sqrt{\frac{k_0 + m^*}{2\, m^*}} \qquad B_{\boldsymbol{k}} = \frac{1}{k_0 + m^*}$$

$$A_k^2\left(1 - \boldsymbol{k}^2\, B_{\boldsymbol{k}}^2\right) = 1 \qquad A_k^2\left(1 + \boldsymbol{k}^2\, B_{\boldsymbol{k}}^2\right) = \frac{k_0}{m^*}$$

$$u_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} \chi_\sigma \\ B_{\boldsymbol{k}}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_\sigma \end{pmatrix} \qquad v_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} B_{\boldsymbol{k}}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$\bar{u}_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(\chi_\sigma^T, -B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\right) \qquad \bar{v}_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}, -\chi_\sigma^T\right)$$

$$u_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(\chi_\sigma^T, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\right) = \bar{u}_\sigma(-\boldsymbol{k}) \qquad v_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}} \left(B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}, \chi_\sigma^T\right) = -\bar{v}_\sigma(-\boldsymbol{k})$$

$$\bar{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'} \qquad \bar{v}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = -\delta_{\sigma\sigma'}$$

$$u_\sigma^+(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*}\, \delta_{\sigma\sigma'} \qquad v_\sigma^+(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*}\, \delta_{\sigma\sigma'}$$

$$\bar{u}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad \bar{v}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = 0$$

$$u_\sigma^+(-\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad v_\sigma^+(-\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = 0$$