6.3. Canonical Quantization

$$\mathcal{L} = c\,\overline{\psi}\left(i\,\hbar\,\gamma^{\mu}\,\partial_{\mu} - m\,c \right)\psi$$

$$= c\,\hbar\,\overline{\psi}\left(i\,\gamma^{\mu}\,\partial_{\mu} - m^{*} \right)\psi \qquad\qquad m^{*} = \frac{m\,c}{\hbar}$$

$$\rightarrow \qquad \pi = \frac{\partial\,\mathcal{L}}{\partial\,\dot{\psi}} = i\,\hbar\,\overline{\psi}\,\gamma^{0} = i\,\hbar\,\psi^{+}$$

$$\overline{\pi} = \frac{\partial\,\mathcal{L}}{\partial\,\dot{\overline{\psi}}} = 0$$

$$\mathcal{H} = i\,\hbar\,\overline{\psi}\,\gamma^{0}\,\dot{\psi} - \mathcal{L}$$

$$= c\,\hbar\,\overline{\psi}\left(-i\,\gamma^{i}\,\partial_{i} + m^{*} \right)\psi$$

$$= c\,\hbar\,\overline{\psi}\left(-i\,\boldsymbol{\gamma}\cdot\nabla + m^{*} \right)\psi$$

$$\rightarrow \qquad H = c\,\hbar \int d^{3}r\;\overline{\psi}\left(-i\,\boldsymbol{\gamma}\cdot\nabla + m^{*} \right)\psi$$

Plane wave expansion (cf. K-G field) :

$$\psi(x) = \sum_{\sigma}\int \frac{d^{3}k}{(2\,\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\left[c_{\sigma}(\boldsymbol{k})\,u_{\sigma}(\boldsymbol{k})\,e^{-i\,k\cdot x} + d_{\sigma}^{+}(\boldsymbol{k})\,v_{\sigma}(\boldsymbol{k})\,e^{i\,k\cdot x} \right]$$

where $\alpha_{\boldsymbol{k}}$ is a real normalization constant to be determined.

As in the K-G case,

$c_{\sigma}(\boldsymbol{k})$ kills a particle with positive energy $E = \hbar\omega_{\boldsymbol{k}} \ge 0$.

$d_{\sigma}^{+}(\boldsymbol{k})$ creates an anti-particle with positive energy $E = \hbar\omega_{\boldsymbol{k}} \ge 0$.

Note: Actually, $d_{\sigma}^{+}(\boldsymbol{k})$ represents the interpretation of a particle with energy $-\hbar\omega_{\boldsymbol{k}}$, spin σ & momentum $\hbar\,\boldsymbol{k}$ as an antiparticle with energy $\hbar\omega_{\boldsymbol{k}}$, spin $-\sigma$ & momentum $-\hbar\,\boldsymbol{k}$. (See for example, the section on Dirac sea in Ezawa's book.)

$$\rightarrow \qquad \psi^{+}(x) = \sum_{\sigma}\int \frac{d^{3}k}{(2\,\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\left[c_{\sigma}^{+}(\boldsymbol{k})\,u_{\sigma}^{+}(\boldsymbol{k})\,e^{i\,k\cdot x} + d_{\sigma}(\boldsymbol{k})\,v_{\sigma}^{+}(\boldsymbol{k})\,e^{-i\,k\cdot x} \right]$$

Note: c_{σ} is a scalar operator so $c_{\sigma}\,u_{\sigma} = u_{\sigma}\,c_{\sigma}$, & similarly for c_{σ}^{+}, d_{σ} & d_{σ}^{+}.

$$\therefore \qquad \overline{\psi}(x) = \sum_{\sigma}\int \frac{d^{3}k}{(2\,\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\left[c_{\sigma}^{+}(\boldsymbol{k})\,\overline{u}_{\sigma}(\boldsymbol{k})\,e^{i\,k\cdot x} + d_{\sigma}(\boldsymbol{k})\,\overline{v}_{\sigma}(\boldsymbol{k})\,e^{-i\,k\cdot x} \right]$$

$$\left(-i\,\boldsymbol{\gamma}\cdot\nabla + m^{*} \right)\psi(x) = \sum_{\sigma}\int \frac{d^{3}k}{(2\,\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\Big[c_{\sigma}(\boldsymbol{k})\left(\boldsymbol{\gamma}\cdot\boldsymbol{k} + m^{*} \right)u_{\sigma}(\boldsymbol{k})\,e^{-i\,k\cdot x}$$

$$+ d_{\sigma}^{+}(\boldsymbol{k})\left(-\boldsymbol{\gamma}\cdot\boldsymbol{k} + m^{*} \right)v_{\sigma}(\boldsymbol{k})\,e^{i\,k\cdot x} \Big]$$

Plane Wave Expansion of H

In the evaluation of H,

$$\int d^{3}r \quad\rightarrow\quad (2\,\pi)^{3}\,\delta(\boldsymbol{k} - \boldsymbol{k}') \qquad \text{for } c_{\sigma}^{+}\,c_{\sigma'} \ \&\ d_{\sigma}\,d_{\sigma'}^{+} \text{ terms}$$

$$(2\,\pi)^{3}\,\delta(\boldsymbol{k} + \boldsymbol{k}') \qquad \text{for } c_{\sigma}^{+}\,d_{\sigma'}^{+} \ \&\ c_{\sigma}\,d_{\sigma'} \text{ terms}$$

so that

$$H = c\hbar \sum_{\sigma\,\sigma'} \int d^3 k \; [\; \alpha_{\boldsymbol{k}}^2\; c_\sigma^+(\boldsymbol{k})\, c_{\sigma'}(\boldsymbol{k})\, \overline{u}_\sigma(\boldsymbol{k})\, (\, \boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, u_{\sigma'}(\boldsymbol{k})$$

$$+\, \alpha_{\boldsymbol{k}}^2\, d_\sigma(\boldsymbol{k})\, d_{\sigma'}^+(\boldsymbol{k})\, \overline{v}_\sigma(\boldsymbol{k})\, (\, -\boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, v_{\sigma'}(\boldsymbol{k})$$

$$+\; \alpha_{-\boldsymbol{k}}\, \alpha_{\boldsymbol{k}}\, c_\sigma^+(-\boldsymbol{k})\, d_{\sigma'}^+(\boldsymbol{k})\, e^{2\,i\,\omega_k\,t}\, \overline{u}_\sigma(-\boldsymbol{k})\, (\, -\boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, v_{\sigma'}(\boldsymbol{k})$$

$$+\; \alpha_{-\boldsymbol{k}}\, \alpha_{\boldsymbol{k}}\, d_\sigma(-\boldsymbol{k})\, c_{\sigma'}(\boldsymbol{k})\, e^{-2\,i\,\omega_k\,t}\, \overline{v}_\sigma(-\boldsymbol{k})\, (\, \boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, u_{\sigma'}(\boldsymbol{k})\,]$$

Using (see 6.2._PlaneWaveSolutions.pdf)

$$A_{\boldsymbol{k}} = \sqrt{\frac{k_0 + m^*}{2\,m^*}} \qquad\qquad B_{\boldsymbol{k}} = \frac{1}{k_0 + m^*}$$

$$A_k^2\left(1 - \boldsymbol{k}^2\, B_{\boldsymbol{k}}^2\right) = 1 \qquad\qquad A_k^2\left(1 + \boldsymbol{k}^2\, B_{\boldsymbol{k}}^2\right) = \frac{k_0}{m^*} \qquad\qquad 2\, A_{\boldsymbol{k}}^2\, B_{\boldsymbol{k}} = \frac{1}{m^*}$$

$$u_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} \chi_\sigma \\ B_{\boldsymbol{k}}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_\sigma \end{pmatrix} \qquad\qquad v_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} B_{\boldsymbol{k}}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$\overline{u}_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}}\, (\; \chi_\sigma^T\,,\, -B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,) \qquad\qquad \overline{v}_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}}\, (\, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,,\, -\chi_\sigma^T\,)$$

$$u_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}}\, (\; \chi_\sigma^T\,,\, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,) \qquad\qquad v_\sigma^+(\boldsymbol{k}) = A_{\boldsymbol{k}}\, (\, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,,\, \chi_\sigma^T\,)$$

$$= \overline{u}_\sigma(-\boldsymbol{k}) \qquad\qquad\qquad = -\overline{v}_\sigma(-\boldsymbol{k})$$

$$\overline{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'} \qquad\qquad \overline{v}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = -\delta_{\sigma\sigma'}$$

$$\overline{u}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = 0 \qquad\qquad \overline{v}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = 0$$

&

$$\boldsymbol{\gamma}\cdot\boldsymbol{k} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix} \qquad\qquad (\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 = \boldsymbol{k}^2$$

we have,

$$\boldsymbol{\gamma}\cdot\boldsymbol{k}\; u_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_\sigma \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_\sigma \end{pmatrix} \qquad\qquad \boldsymbol{\gamma}\cdot\boldsymbol{k}\;\; v_\sigma(\boldsymbol{k}) = A_{\boldsymbol{k}} \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_\sigma \\ -B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_\sigma \end{pmatrix}$$

$$\therefore \qquad \overline{u}_\sigma(\boldsymbol{k})\, \boldsymbol{\gamma}\cdot\boldsymbol{k}\; u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2\, (\; \chi_\sigma^T\,,\, -B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,) \begin{pmatrix} B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_{\sigma'} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_{\sigma'} \end{pmatrix}$$

$$= 2\, A_{\boldsymbol{k}}^2\, B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_\sigma^T\, \chi_{\sigma'}$$

$$= \frac{\boldsymbol{k}^2}{m^*}\, \delta_{\sigma\sigma'}$$

$$m^*\, \overline{u}_\sigma(\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'}\, m^*$$

$$\rightarrow \qquad \overline{u}_\sigma(\boldsymbol{k})\, (\, \boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, u_{\sigma'}(\boldsymbol{k}) = \left(\frac{\boldsymbol{k}^2}{m^*} + m^* \right) \delta_{\sigma\sigma'}$$

$$-\overline{v}_\sigma(\boldsymbol{k})\, \boldsymbol{\gamma}\cdot\boldsymbol{k} v_{\sigma'}(\boldsymbol{k}) = -A_{\boldsymbol{k}}^2\, (\, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,,\, -\chi_\sigma^T\,) \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_{\sigma'} \\ -B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_{\sigma'} \end{pmatrix}$$

$$= -2\, A_{\boldsymbol{k}}^2\, B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_\sigma^T\, \chi_{\sigma'} = -\frac{\boldsymbol{k}^2}{m^*}\, \delta_{\sigma\sigma'}$$

$$m^*\, \overline{v}_\sigma(\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = -\delta_{\sigma\sigma'}\, m^*$$

$$\rightarrow \qquad \overline{v}_\sigma(\boldsymbol{k})\, (\, -\boldsymbol{\gamma}\cdot\boldsymbol{k} \;+\; m^*\,)\, v_{\sigma'}(\boldsymbol{k}) = -\left(\frac{\boldsymbol{k}^2}{m^*} + m^* \right) \delta_{\sigma\sigma'}$$

$$-\overline{u}_\sigma(-\boldsymbol{k})\, \boldsymbol{\gamma}\cdot\boldsymbol{k}\; v_{\sigma'}(\boldsymbol{k}) = -A_{\boldsymbol{k}}^2\, (\; \chi_\sigma^T\,,\, B_{\boldsymbol{k}}\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\,) \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\;\; \chi_{\sigma'} \\ -B_{\boldsymbol{k}}\, \boldsymbol{k}^2\, \chi_{\sigma'} \end{pmatrix}$$

$$= -\, A_{\boldsymbol{k}}^2\, \left(1 - B_{\boldsymbol{k}}^2\, \boldsymbol{k}^2\right) \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\; \chi_{\sigma'}$$

$$= -\, \chi_\sigma^T\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\; \chi_{\sigma'}$$

$$m^* \bar{u}_\sigma(-\boldsymbol{k})\, v_{\sigma'}(\boldsymbol{k}) = m^* A_{\boldsymbol{k}}^2 \left(\chi_\sigma^T ,\, B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k} \right) \begin{pmatrix} B_{\boldsymbol{k}} \boldsymbol{\sigma}\cdot\boldsymbol{k}\ \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$= 2\, m^* A_{\boldsymbol{k}}^2 B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$= \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$\rightarrow \qquad \bar{u}_\sigma(-\boldsymbol{k}) \left(-\boldsymbol{\gamma}\cdot\boldsymbol{k} + m^* \right) v_{\sigma'}(\boldsymbol{k}) = 0$$

$$\bar{v}_\sigma(-\boldsymbol{k})\, \boldsymbol{\gamma}\cdot\boldsymbol{k}\, u_{\sigma'}(\boldsymbol{k}) = A_{\boldsymbol{k}}^2 \left(-B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k} ,\, -\chi_\sigma^T \right) \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k}\ \chi_{\sigma'} \\ -B_{\boldsymbol{k}} \boldsymbol{k}^2 \chi_{\sigma'} \end{pmatrix}$$

$$= -A_{\boldsymbol{k}}^2 \left(1 - B_{\boldsymbol{k}}^2 \boldsymbol{k}^2 \right) \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$= -\chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$m^* \bar{v}_\sigma(-\boldsymbol{k})\, u_{\sigma'}(\boldsymbol{k}) = m^* A_{\boldsymbol{k}}^2 \left(\chi_\sigma^T ,\, B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k} \right) \begin{pmatrix} B_{\boldsymbol{k}} \boldsymbol{\sigma}\cdot\boldsymbol{k}\ \chi_\sigma \\ \chi_\sigma \end{pmatrix}$$

$$= 2\, m^* A_{\boldsymbol{k}}^2 B_{\boldsymbol{k}} \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$= \chi_\sigma^T \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_{\sigma'}$$

$$\rightarrow \qquad \bar{v}_\sigma(-\boldsymbol{k}) \left(\boldsymbol{\gamma}\cdot\boldsymbol{k} + m^* \right) u_{\sigma'}(\boldsymbol{k}) = 0$$

$$\therefore \qquad H = c\hbar \sum_\sigma \int d^3k\ \alpha_{\boldsymbol{k}}^2 \left(\frac{\boldsymbol{k}^2}{m^*} + m^* \right) \left[c_\sigma^+(\boldsymbol{k})\, c_\sigma(\boldsymbol{k}) - d_\sigma(\boldsymbol{k})\, d_\sigma^+(\boldsymbol{k}) \right]$$

$$= \sum_\sigma \int d^3k\ \alpha_{\boldsymbol{k}}^2 \frac{\hbar}{m^* c} \omega_{\boldsymbol{k}}^2 \left[c_\sigma^+(\boldsymbol{k})\, c_\sigma(\boldsymbol{k}) - d_\sigma(\boldsymbol{k})\, d_\sigma^+(\boldsymbol{k}) \right]$$

where $\left(\frac{\omega_{\boldsymbol{k}}}{c} \right)^2 = \boldsymbol{k}^2 + m^{*2}$

Setting

$$\alpha_{\boldsymbol{k}}^2 = \frac{m^* c}{\omega_{\boldsymbol{k}}} = \frac{m c^2}{\hbar\, \omega_{\boldsymbol{k}}}$$

we have

$$H = \sum_\sigma \int d^3k\ \hbar\, \omega_{\boldsymbol{k}} \left[c_\sigma^+(\boldsymbol{k})\, c_\sigma(\boldsymbol{k}) - d_\sigma(\boldsymbol{k})\, d_\sigma^+(\boldsymbol{k}) \right]$$

&

$$\psi(x) = \sum_\sigma \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}} \left[c_\sigma(\boldsymbol{k})\, u_\sigma(\boldsymbol{k})\, e^{-i k\cdot x} + d_\sigma^+(\boldsymbol{k})\, v_\sigma(\boldsymbol{k})\, e^{i k\cdot x} \right]$$

$$\bar{\psi}(x) = \sum_\sigma \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}} \left[c_\sigma^+(\boldsymbol{k})\, \bar{u}_\sigma(\boldsymbol{k})\, e^{i k\cdot x} + d_\sigma(\boldsymbol{k})\, \bar{v}_\sigma(\boldsymbol{k})\, e^{-i k\cdot x} \right]$$

Quantization

If we quantize via the commutator

$$\left[d_\sigma(\boldsymbol{k}),\ d_{\sigma'}^+(\boldsymbol{k}') \right] = \delta_{\sigma\sigma'}\, \delta(\boldsymbol{k} - \boldsymbol{k}')$$

then

$$-d_\sigma(\boldsymbol{k})\, d_\sigma^+(\boldsymbol{k}) = -d_\sigma^+(\boldsymbol{k})\, d_\sigma(\boldsymbol{k}) - \delta_{\sigma\sigma'}\, \delta(0)$$

whose expectation value is always negative since the number of antiparticles $\left\langle d_\sigma^+(\boldsymbol{k})\, d_\sigma(\boldsymbol{k}) \right\rangle$ must be positive. Thus, $\langle H \rangle$ can be negative, which contradicts with our specific construction of ψ so that it deals only with particles & antipartcles of positive energies.

To avoid the disaster, we quantize via the anticommutator so that

$$\left\{ c_\sigma(\boldsymbol{k}),\ c_{\sigma'}^+(\boldsymbol{k}') \right\} = \delta_{\sigma\sigma'}\, \delta(\boldsymbol{k} - \boldsymbol{k}')$$

$$\{d_\sigma(\boldsymbol{k}),\ d^+_{\sigma'}(\boldsymbol{k}')\} = \delta_{\sigma\sigma'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')$$
$$\{c_\sigma(\boldsymbol{k}),\ c_{\sigma'}(\boldsymbol{k}')\} = \{c^+_\sigma(\boldsymbol{k}),\ c^+_{\sigma'}(\boldsymbol{k}')\} = 0$$

→ $$H = \sum_\sigma \int d^3k\ \hbar\,\omega_{\boldsymbol{k}}\,[c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k}) + d^+_\sigma(\boldsymbol{k})\,d_\sigma(\boldsymbol{k})] - \delta(0)\sum_\sigma \int d^3k\ \hbar\,\omega_{\boldsymbol{k}}$$

or $$:H: = \sum_\sigma \int d^3k\ \hbar\,\omega_{\boldsymbol{k}}\,[c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k}) + d^+_\sigma(\boldsymbol{k})\,d_\sigma(\boldsymbol{k})]$$

Backtracking to the real space, we have

$$\{\psi_\sigma(t,\boldsymbol{r}),\ \pi_{\sigma'}(t,\boldsymbol{r}')\} = i\,\hbar\,\delta_{\sigma\sigma'}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$
$$\{\psi_\sigma(t,\boldsymbol{r}),\ \psi^+_{\sigma'}(t,\boldsymbol{r}')\} = \delta_{\sigma\sigma'}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

&

$$\{\psi_\sigma(t,\boldsymbol{r}),\ \psi_{\sigma'}(t,\boldsymbol{r}')\} = \{\psi^+_\sigma(t,\boldsymbol{r}),\ \psi^+_{\sigma'}(t,\boldsymbol{r}')\} = 0$$

P'

Using

$$\overline{\psi}(x) = \sum_\sigma \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}}\ [c^+_\sigma(\boldsymbol{k})\,\overline{u}_\sigma(\boldsymbol{k})\,e^{i\,k\cdot x} + d_\sigma(\boldsymbol{k})\,\overline{v}_\sigma(\boldsymbol{k})\,e^{-i\,k\cdot x}]$$

$$\frac{\hbar}{i}\nabla\psi(x) = \sum_\sigma \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}}\ \hbar\,\boldsymbol{k}\,[c_\sigma(\boldsymbol{k})\,u_\sigma(\boldsymbol{k})\,e^{-i\,k\cdot x} - d^+_\sigma(\boldsymbol{k})\,v_\sigma(\boldsymbol{k})\,e^{i\,k\cdot x}]$$

&

$$\overline{u}_\sigma(\boldsymbol{k})\,u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'} \qquad \overline{v}_\sigma(\boldsymbol{k})\,v_{\sigma'}(\boldsymbol{k}) = -\delta_{\sigma\sigma'}$$
$$\overline{u}_\sigma(\boldsymbol{k})\,v_{\sigma'}(\boldsymbol{k}) = 0 \qquad \overline{v}_\sigma(\boldsymbol{k})\,u_{\sigma'}(\boldsymbol{k}) = \delta_{\sigma\sigma'}$$

we have

$$\boldsymbol{P}' = \int d^3r\,\overline{\psi}\,\frac{\hbar}{i}\nabla\psi(x)$$
$$= \sum_\sigma \int d^3k\,\frac{m^* c}{\omega_{\boldsymbol{k}}}\,\hbar\,\boldsymbol{k}\,[c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k}) + d_\sigma(\boldsymbol{k})\,d^+_\sigma(\boldsymbol{k})]$$

Using

$$\delta(0)\int d^3k\ \hbar\,\boldsymbol{k} = 0$$

we have

$$\boldsymbol{P}' = \sum_\sigma \int d^3k\,\frac{m^* c}{\omega_{\boldsymbol{k}}}\,\hbar\,\boldsymbol{k}\,[c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k}) - d^+_\sigma(\boldsymbol{k})\,d_\sigma(\boldsymbol{k})]$$

Since $d^+_\sigma(\boldsymbol{k})$ creates antiparticle of momentum $\hbar\,\boldsymbol{k}$, we expect the total momentum to be

$$\boldsymbol{P} = \sum_\sigma \int d^3k\,\hbar\,\boldsymbol{k}\,[c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k}) + d^+_\sigma(\boldsymbol{k})\,d_\sigma(\boldsymbol{k})] \ \neq \boldsymbol{P}'$$

P

Since $\mathcal{P}^\mu = (\mathcal{H}/c, \boldsymbol{\mathcal{P}})$ is a 4-vector, we expect $\mathcal{P}^\mu \propto \overline{\psi}\,\gamma^\mu\,\psi$.
As a quantum operator, we expect $\mathcal{P}^\mu \propto i\,\hbar\,\partial^\mu$.
Reconciling these two expectations requires us to define

$$\mathcal{P}^\mu = \mathcal{T}^{\mu 0} = \mathcal{T}^{0\mu}$$

where

$$\mathcal{T}^{\mu\nu} = i\,\hbar\,\overline{\psi}\,\gamma^\mu\,\partial^\nu\,\psi$$

is the energy-momentum density tensor.

Thus,

$$T^{\mu}_{\nu} = \int d^3 r \, \mathcal{T}^{\mu}_{\nu} = i\hbar \int d^3 r \, \overline{\psi} \gamma^{\mu} \partial_{\nu} \psi$$

We're only interested in

$$P_{\mu} = i\hbar \int d^3 r \, \overline{\psi} \gamma^0 \partial_{\mu} \psi = i\hbar \int d^3 r \, \psi^+ \partial_{\mu} \psi$$

$$\psi(x) = \sum_{\sigma} \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}} \left[c_{\sigma}(\boldsymbol{k}) u_{\sigma}(\boldsymbol{k}) e^{-ik\cdot x} + d^+_{\sigma}(\boldsymbol{k}) v_{\sigma}(\boldsymbol{k}) e^{ik\cdot x} \right]$$

$$\rightarrow \qquad \psi^+(x) = \sum_{\sigma} \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}} \left[c^+_{\sigma}(\boldsymbol{k}) u^+_{\sigma}(\boldsymbol{k}) e^{ik\cdot x} + d_{\sigma}(\boldsymbol{k}) v^+_{\sigma}(\boldsymbol{k}) e^{-ik\cdot x} \right]$$

$$= \sum_{\sigma} \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}} \left[c^+_{\sigma}(\boldsymbol{k}) u_{\sigma}(\boldsymbol{k}) e^{ik\cdot x} + d_{\sigma}(\boldsymbol{k}) v^+_{\sigma}(\boldsymbol{k}) e^{-ik\cdot x} \right]$$

$$i\hbar \partial_{\mu} \psi(x) = \sum_{\sigma} \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}} \hbar k_{\mu} \left[c_{\sigma}(\boldsymbol{k}) u_{\sigma}(\boldsymbol{k}) e^{-ik\cdot x} - d^+_{\sigma}(\boldsymbol{k}) v_{\sigma}(\boldsymbol{k}) e^{ik\cdot x} \right]$$

As in the case for H, after $\int d^3 r$, we have

$$\begin{aligned} P_{\mu} = \sum_{\sigma\sigma'} \int d^3 k \, \alpha^2_{\boldsymbol{k}} \, \hbar k_{\mu} \big[& c^+_{\sigma}(\boldsymbol{k}) c_{\sigma'}(\boldsymbol{k}) \, u^+_{\sigma}(\boldsymbol{k}) u_{\sigma'}(\boldsymbol{k}) \\ & - d_{\sigma}(\boldsymbol{k}) d^+_{\sigma'}(\boldsymbol{k}) v^+_{\sigma} v_{\sigma'}(\boldsymbol{k}) \\ & - c^+_{\sigma}(-\boldsymbol{k}) d^+_{\sigma'}(\boldsymbol{k}) e^{2i\omega_{\boldsymbol{k}} t} u^+_{\sigma}(-\boldsymbol{k}) v_{\sigma'}(\boldsymbol{k}) \\ & + d_{\sigma}(-\boldsymbol{k}) c_{\sigma'}(\boldsymbol{k}) e^{-2i\omega_{\boldsymbol{k}} t} v^+_{\sigma\sigma}(-\boldsymbol{k}) u_{\sigma'}(\boldsymbol{k}) \big] \end{aligned}$$

Using

$$u^+_{\sigma}(\boldsymbol{k}) u_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*} \delta_{\sigma\sigma'} \qquad v^+_{\sigma}(\boldsymbol{k}) v_{\sigma'}(\boldsymbol{k}) = \frac{k_0}{m^*} \delta_{\sigma\sigma'}$$

$$u^+_{\sigma}(-\boldsymbol{k}) v_{\sigma'}(\boldsymbol{k}) = 0 \qquad v^+_{\sigma}(-\boldsymbol{k}) u_{\sigma'}(\boldsymbol{k}) = 0$$

$$\alpha^2_{\boldsymbol{k}} = \frac{m^* c}{\omega_{\boldsymbol{k}}}$$

we have

$$P_{\mu} = \sum_{\sigma} \int d^3 k \, \alpha^2_{\boldsymbol{k}} \frac{k_0}{m^*} \hbar k_{\mu} \left[c^+_{\sigma}(\boldsymbol{k}) c_{\sigma'}(\boldsymbol{k}) - d_{\sigma}(\boldsymbol{k}) d^+_{\sigma'}(\boldsymbol{k}) \right]$$

$$= \sum_{\sigma} \int d^3 k \, \hbar k_{\mu} \left[c^+_{\sigma}(\boldsymbol{k}) c_{\sigma}(\boldsymbol{k}) + d^+_{\sigma}(\boldsymbol{k}) d_{\sigma}(\boldsymbol{k}) \right]$$

as promised. There's no zero-point term because

$$\int d^3 k \, k_{\mu} = -\int d^3 k \, k_{\mu} = 0$$

Noether's Current

$$\mathcal{L} = c\,\overline{\psi} \left(i\hbar \gamma^{\mu} \partial_{\mu} - mc \right) \psi$$

is invariant under a global phase transformation

$$\psi \rightarrow e^{i\epsilon} \psi \qquad \overline{\psi} \rightarrow \overline{\psi} e^{-i\epsilon}$$

$$\therefore \qquad \delta\psi \approx i\epsilon\psi \qquad \delta\overline{\psi} \approx -i\epsilon\overline{\psi}$$

The corresponding Noether current is therefore (see 3.5._NoetherCurrents.pdf)

$$j^\mu = i\,\frac{\partial \mathcal{L}}{\partial\,\partial_\mu \psi}\,\psi - i\,\frac{\partial \mathcal{L}}{\partial\,\partial_\mu \overline{\psi}}\,\overline{\psi}$$
$$= -\hbar\,c\,\overline{\psi}\,\gamma^\mu\,\psi$$

Thus,

$$\partial_\mu j^\mu = -\hbar\,c\,\overline{\psi}\,\gamma^\mu\left(\overleftarrow{\partial}_\mu + \partial_\mu\right)\psi \;= -\frac{c}{i}\,\overline{\psi}\,(-m\,c + m\,c)\,\psi \;= 0$$

as expected.

The conserved Noether charge is

$$Q = \frac{1}{c}\int d^3 r\, j^0 \;= -\hbar\int d^3 r\,\overline{\psi}\,\gamma^0\,\psi \;= -\hbar\int d^3 r\,\psi^+\,\psi$$

The integral is analogous to that for P_μ. Without ∂_μ, there's no k_μ nor sign flip for the $d_\sigma(\boldsymbol{k})\,d^+_{\sigma'}(\boldsymbol{k})$ term. Hence,

$$Q = -\hbar\sum_\sigma\int d^3 k\;\left[\,c^+_\sigma(\boldsymbol{k})\,c_\sigma(\boldsymbol{k})\; - d^+_\sigma(\boldsymbol{k})\,d_\sigma(\boldsymbol{k})\,\right]$$

If Q is interpreted as proportional to the (conserved) total charge of the system, then particle & antiparticle must have opposite charges.