6.3.a. Fermi Sea

The interpretation of a negative energy particle state as an anti-particle state of positive energy was already discussed in Chap. 3 for the K-G (bosonic) fields. The same procedure is also applicable to the Dirac field.

Historically, before the concept of anti-particles, Dirac uses the Fermi sea to justify such an interpretation. The hole concept was subsequently abandoned by particle physicists as being unnecessarily complicated while the anti-particles themselves were experimentally observed. However, in semi-conductor physics, the hole concept is the preferred one since it is closer to the actual physics.

Retracing the steps followed in 3.3._RealKlein-GordonField.pdf, we begin with

$$\psi(x) = \sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}} \left[c_\sigma^{(+)}(\boldsymbol{k})\, u_\sigma^{(+)}(\boldsymbol{k})\, e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} + c_\sigma^{(-)}(\boldsymbol{k})\, u_\sigma^{(-)}(\boldsymbol{k})\, e^{i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} \right]$$

where $\alpha_{\boldsymbol{k}} = \sqrt{\dfrac{m^* c}{\omega_{\boldsymbol{k}}}} = \dfrac{m c^2}{\hbar\,\omega_{\boldsymbol{k}}}$

& the superscript $+/-$ denotes the sign of the energy of the plane wave.

Note that both $c_\sigma^{(\pm)}(\boldsymbol{k})$ are annihilation operators of plane waves of energies $\pm\hbar\,\omega_{\boldsymbol{k}}$ but momenta $+\hbar\,\boldsymbol{k}$.

Thus, after the usual rigmarole (c.f. 6.3._CanonicalQuantization.pdf), we should get

$$H = \sum_\sigma \int d^3 k\ \hbar\,\omega_{\boldsymbol{k}} \left[c_\sigma^{(+)+}(\boldsymbol{k})\, c_\sigma^{(+)}(\boldsymbol{k}) - c_\sigma^{(-)+}(\boldsymbol{k})\, c_\sigma^{(-)}(\boldsymbol{k}) \right]$$

$$\boldsymbol{P} = \sum_\sigma \int d^3 k\ \hbar\,\boldsymbol{k} \left[c_\sigma^{(+)+}(\boldsymbol{k})\, c_\sigma^{(+)}(\boldsymbol{k}) + c_\sigma^{(-)+}(\boldsymbol{k})\, c_\sigma^{(-)}(\boldsymbol{k}) \right]$$

Taking $\boldsymbol{k} \to -\boldsymbol{k}$, the $(-)$ term in ψ becomes

$$\sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}}\, c_\sigma^{(-)}(-\boldsymbol{k})\, u_\sigma^{(-)}(-\boldsymbol{k})\, e^{i k\cdot x}$$

Thus, setting

$$d_\sigma^+(\boldsymbol{k}) = c_{-\sigma}^{(-)}(-\boldsymbol{k}) \qquad\qquad v_\sigma(\boldsymbol{k}) = u_{-\sigma}^{(-)}(-\boldsymbol{k})$$

where $\sigma = \pm 1$ is used to denote spin $\uparrow / \downarrow$, turns it into

$$\sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}}\, d_{-\sigma}^+(\boldsymbol{k})\, v_{-\sigma}(\boldsymbol{k})\, e^{i k\cdot x} = \sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \alpha_{\boldsymbol{k}}\, d_\sigma^+(\boldsymbol{k})\, v_\sigma(\boldsymbol{k})\, e^{i k\cdot x}$$

the integrand of which describes creating a plane wave of positive energy $\hbar\,\omega_{\boldsymbol{k}}$ & momentum $\hbar\,\boldsymbol{k}$.

The Fermi (or Dirac) sea is a Dirac field with all the negative energy states filled.
This is possible since according to the Pauli exclusion principle, each state can accomodate just 1 particle. Since for each state with momentum $\boldsymbol{p}$, there is a state with $-\boldsymbol{p}$, & likewise for the spins, we have

$$\boldsymbol{P} = 0 \qquad \& \qquad \boldsymbol{S} = 0$$

On the other hand,

$$E = E_0 = -\infty$$

Thus, when $c_{-\sigma}^{(-)}(-\boldsymbol{k})$ annihilates a particle in the state of momentum $-\hbar\,\boldsymbol{k}$, spin $-\sigma$ & energy $-\hbar\omega_{\boldsymbol{k}}$ in the Fermi sea, a hole is created while the system is left with

$$\boldsymbol{P} = 0 - (-\hbar\,\boldsymbol{k}) = \hbar\,\boldsymbol{k} \quad , \qquad \boldsymbol{S} = 0 - \left(-\frac{1}{2}\sigma\right) = \frac{1}{2}\sigma$$

& $\quad E - E_0 = -(-\hbar\omega_{\boldsymbol{k}}) = \hbar\omega_{\boldsymbol{k}}$

One can assign these attributes either to the hole or to an anti-particle so that $d_\sigma^+(\boldsymbol{k})$ creates. The advantage of the anti-particle interpretation is that we can invoke the renormalization scheme & drop the 'zero-point energy' E_0. In doing do, we've also replace the Fermi sea with the vacuum.