6.4. Interaction with EM Fields

Minimal coupling:

$$p^\mu \quad \to \quad p^\mu - \frac{q}{c} A^\mu$$

$$\partial_\mu \quad \to \quad \partial_\mu + i \frac{q}{\hbar c} A_\mu = \partial_\mu + i q^* A_\mu \qquad \left(q^* = \frac{q}{\hbar c} \right)$$

$$\to \qquad \mathcal{L} = \mathcal{L}_M + \mathcal{L}_{\text{EM}}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_M = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}}$$

$$= c\hbar \overline{\psi} \left[i \gamma^\mu (\partial_\mu + i q^* A_\mu) - m^* \right] \psi \qquad \left(m^* = \frac{mc}{\hbar} \right)$$

$$\to \qquad \mathcal{L}_{\text{int}} = -c\hbar q^* \overline{\psi} \gamma^\mu A_\mu \psi$$

$$\pi = \frac{\partial \mathcal{L}_M}{\partial \dot{\psi}} = i\hbar \overline{\psi} \gamma^0 = i\hbar \psi^+$$

$$\overline{\pi} = \frac{\partial \mathcal{L}_M}{\partial \dot{\overline{\psi}}} = 0$$

$$\mathcal{H}_M = i\hbar \overline{\psi} \gamma^0 \dot{\psi} - \mathcal{L}_M$$

$$= c\hbar \overline{\psi} \left(-i \gamma^i \partial_i + q^* \gamma^\mu A_\mu + m^* \right) \psi$$

$$= c\hbar \overline{\psi} \left[-i \gamma^i (\partial_i + i q^* A_i) + q^* \gamma^0 \varphi + m^* \right] \psi$$

$$= c\hbar \overline{\psi} \left[-i \boldsymbol{\gamma} \cdot (\nabla - i q^* \mathbf{A}) + q^* \gamma^0 \varphi + m^* \right] \psi$$

$$\to \qquad \mathcal{H}_{\text{int}} = c\hbar q^* \overline{\psi} \gamma^\mu A_\mu \psi$$

Euler eq.

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} = -\frac{1}{4\pi} \partial_\nu F^{\nu\mu}$$

Comparing with the Maxwell eq.

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J_q^\mu$$

we have

$$J_q^\mu = -c \frac{\partial \mathcal{L}}{\partial A_\mu}$$

Charge current density:

$$J_q^\mu = -c \frac{\partial \mathcal{L}}{\partial A_\mu} = -c \frac{\partial \mathcal{L}_M}{\partial A_\mu} = -c \frac{\partial \mathcal{L}_{\text{int}}}{\partial A_\mu}$$

$$= c^2 \hbar q^* \overline{\psi} \gamma^\mu \psi = -q^* c J^\mu_{\text{Noether}}$$

$$= q c \overline{\psi} \gamma^\mu \psi = -\frac{q}{\hbar} J^\mu_{\text{Noether}}$$

Thus, the total charge is

$$Q_{\text{EM}} = \frac{1}{c} \int d^3 r \, J_q^0 = -\frac{q}{\hbar} Q_{\text{Noether}}$$

$$= q \sum_\sigma \int d^3 k \; [\, c^+_\sigma(\boldsymbol{k})\, c_\sigma(\boldsymbol{k}) \; - d^+_\sigma(\boldsymbol{k})\, d_\sigma(\boldsymbol{k}) \,]$$

For electrons, $q = -e < 0$.

$\langle \boldsymbol{p}', \sigma' \mid H_{\text{int}} \mid \boldsymbol{p}, \sigma \rangle_{\text{electron}}$

We wish to evaluate the matrix element

$$\begin{aligned} M &= \langle \boldsymbol{p}', \sigma' \mid H_{\text{int}} \mid \boldsymbol{p}, \sigma \rangle \\ &= \int d^3 r \, \langle \boldsymbol{p}', \sigma' \mid \mathcal{H}_{\text{int}} \mid \boldsymbol{p}, \sigma \rangle \\ &= c\, \hbar\, q^* \int d^3 r \, \langle 0 \mid c_{\sigma'}(\boldsymbol{k}')\, \overline{\psi}\, \gamma^\mu A_\mu\, \psi\, c^+_\sigma(\boldsymbol{k}) \mid 0 \rangle \end{aligned}$$

where

$$\mid \boldsymbol{p}, \sigma \rangle = c^+_\sigma(\boldsymbol{k}) \mid 0 \rangle \quad \& \quad \mid \boldsymbol{p}', \sigma' \rangle = c^+_{\sigma'}(\boldsymbol{k}') \mid 0 \rangle$$

are 2 electron states.

Using

$$\psi(x) = \sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}} \left[c_\sigma(\boldsymbol{k})\, u_\sigma(\boldsymbol{k})\, e^{-i k\cdot x} + d^+_\sigma(\boldsymbol{k})\, v_\sigma(\boldsymbol{k})\, e^{i k \cdot x} \right] \Bigg|_{\boldsymbol{k}=\boldsymbol{k}_1,\, \sigma=\sigma_1}$$

$$\overline{\psi}(x) = \sum_\sigma \int \frac{d^3 k}{(2\pi)^{3/2}} \sqrt{\frac{m^* c}{\omega_{\boldsymbol{k}}}} \left[c^+_\sigma(\boldsymbol{k})\, \overline{u}_\sigma(\boldsymbol{k})\, e^{i k\cdot x} + d_\sigma(\boldsymbol{k})\, \overline{v}_\sigma(\boldsymbol{k})\, e^{-i k \cdot x} \right] \Bigg|_{\boldsymbol{k}=\boldsymbol{k}_2,\, \sigma=\sigma_2}$$

$$\langle \boldsymbol{p}', \sigma' \mid \boldsymbol{p}, \sigma \rangle = \delta_{\sigma' \sigma}\, \delta(\boldsymbol{k} - \boldsymbol{k}')$$

$$c \mid 0 \rangle = 0 = d \mid 0 \rangle \qquad \langle 0 \mid c^+ = 0 = \langle 0 \mid d^+$$

we see that, of the 4 terms in the product $\overline{\psi}(x)\, \psi(x)$, only 2 terms survive:

1. $\langle 0 \mid c_{\sigma'}(\boldsymbol{k}')\, c^+_{\sigma_2}(\boldsymbol{k}_2)\, c_{\sigma_1}(\boldsymbol{k}_1)\, c^+_\sigma(\boldsymbol{k}) \mid 0 \rangle = \delta_{\sigma' \sigma_2}\, \delta(\boldsymbol{k}' - \boldsymbol{k}_2)\, \delta_{\sigma \sigma_1}\, \delta(\boldsymbol{k} - \boldsymbol{k}_1)$
2. $\langle 0 \mid c_{\sigma'}(\boldsymbol{k}')\, d_{\sigma_2}(\boldsymbol{k}_2)\, d^+_{\sigma_1}(\boldsymbol{k}_1)\, c^+_\sigma(\boldsymbol{k}) \mid 0 \rangle = \delta_{\sigma' \sigma}\, \delta(\boldsymbol{k} - \boldsymbol{k}')\, \delta_{\sigma_2 \sigma_1}\, \delta(\boldsymbol{k}_2 - \boldsymbol{k}_1)$

After the $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ integration, type 1 contributes to M an amount

$$\begin{aligned} M_1 &= \frac{c\, \hbar\, q^*}{(2\pi)^3} \int d^3 r \, \frac{m^* c}{\sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{r})\, \overline{u}_{\sigma'}(\boldsymbol{k}')\, \gamma^\mu\, u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t + i(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{r}} \\ &= \frac{c\, \hbar\, q^*}{(2\pi)^3} \frac{m^* c}{\sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{k} - \boldsymbol{k}')\, \overline{u}_{\sigma'}(\boldsymbol{k}')\, \gamma^\mu\, u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t} \\ &= \frac{1}{(2\pi)^3} \frac{q\, m\, c^2}{\hbar \sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{k} - \boldsymbol{k}')\, \overline{u}_{\sigma'}(\boldsymbol{k}')\, \gamma^\mu\, u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t} \end{aligned}$$

where

$$A_\mu(\boldsymbol{k}) = \int d^3 r \, A_\mu(\boldsymbol{r})\, e^{i \boldsymbol{k}\cdot \boldsymbol{r}}$$

After the $\boldsymbol{k}_2$ integration, type 2 contributes to M an amount

$$\begin{aligned} M_2 &= \delta_{\sigma' \sigma}\, \delta(\boldsymbol{k} - \boldsymbol{k}') \frac{c\, \hbar\, q^*}{(2\pi)^3} \int d^3 r \, A_\mu(\boldsymbol{r}) \sum_{\sigma_1} \int d^3 k_1 \frac{m^* c}{\omega_{\boldsymbol{k}_1}} \overline{v}_{\sigma_1}(\boldsymbol{k}_1)\, \gamma^\mu\, v_{\sigma_1}(\boldsymbol{k}_1) \\ &= \delta_{\sigma' \sigma}\, \delta(\boldsymbol{k} - \boldsymbol{k}') \frac{c\, \hbar\, q^*}{(2\pi)^3} A_\mu(0)\, G^\mu \end{aligned}$$

where

$$G^\mu = \sum_\sigma \int d^3 k \frac{m^* c}{\omega_{\boldsymbol{k}}} \bar{v}_\sigma(\boldsymbol{k})\, \gamma^\mu\, v_\sigma(\boldsymbol{k})$$

Using

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}$$

& $$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$$

we have

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i\,\sigma^{\mu\nu}$$

$$\rightarrow \quad \gamma^\mu \gamma^\nu k_\nu = k^\mu - i\,\sigma^{\mu\nu} k_\nu$$

$$(\gamma^\mu k_\mu - m^*)\, u_\sigma(\boldsymbol{k}) = 0$$

$$\rightarrow \quad \gamma^\mu \gamma^\nu k_\nu\, u_\sigma(\boldsymbol{k}) - m^* \gamma^\mu u_\sigma(\boldsymbol{k}) = 0$$

$$\therefore \quad \gamma^\mu u_\sigma(\boldsymbol{k}) = \frac{1}{m^*} \gamma^\mu \gamma^\nu k_\nu\, u_\sigma(\boldsymbol{k}) = \frac{1}{m^*}\left(k^\mu - i\,\sigma^{\mu\nu} k_\nu\right) u_\sigma(\boldsymbol{k})$$

Similarly,

$$\gamma^\nu \gamma^\mu = \eta^{\mu\nu} + i\,\sigma^{\mu\nu}$$

$$\rightarrow \quad \gamma^\nu \gamma^\mu k_\nu = k^\mu + i\,\sigma^{\mu\nu} k_\nu$$

$$\bar{u}_\sigma(\boldsymbol{k})\,(\gamma^\mu k_\mu - m^*) = 0$$

$$\rightarrow \quad \bar{u}_\sigma(\boldsymbol{k})\, \gamma^\nu \gamma^\mu k_\nu - m^* \bar{u}_\sigma(\boldsymbol{k})\, \gamma^\mu = 0$$

$$\therefore \quad \bar{u}_\sigma(\boldsymbol{k})\, \gamma^\mu = \frac{1}{m^*} \bar{u}_\sigma(\boldsymbol{k})\, \gamma^\nu \gamma^\mu k_\nu = \frac{1}{m^*} \bar{u}_\sigma(\boldsymbol{k})\left(k^\mu + i\,\sigma^{\mu\nu} k_\nu\right)$$

$$\rightarrow \quad M_1 = \frac{c\,\hbar\, q^*}{(2\pi)^3} \frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{k} - \boldsymbol{k}')\, \bar{u}_{\sigma'}(\boldsymbol{k}')\left(k^\mu - i\,\sigma^{\mu\nu} k_\nu\right) u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t}$$

$$= \frac{c\,\hbar\, q^*}{(2\pi)^3} \frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{k} - \boldsymbol{k}')\, \bar{u}_{\sigma'}(\boldsymbol{k}')\left(k'^\mu + i\,\sigma^{\mu\nu} k'_\nu\right) u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t}$$

$$= \frac{c\,\hbar\, q^*}{(2\pi)^3} \frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\, \omega_{\boldsymbol{k}}}} A_\mu(\boldsymbol{k} - \boldsymbol{k}')\, \bar{u}_{\sigma'}(\boldsymbol{k}')\, \frac{1}{2}\left[k'^\mu + k^\mu + i\,\sigma^{\mu\nu}(k'_\nu - k_\nu)\right] u_\sigma(\boldsymbol{k})\, e^{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})t}$$

Zeeman Effect

For a constant magnetic field,

$$A^\mu = (0, \boldsymbol{A}) \qquad \& \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

In the standard representation of $\{\gamma^\mu\}$ (see 6.1._DiracEquation.pdf):

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_0 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\gamma_i$$

where

$$[\sigma^i, \sigma^j] = 2\,i\,\varepsilon^{ijk} \sigma^k \qquad \{\sigma^i, \sigma^j\} = 2\,\delta^{ij}$$

$$\sigma^i \sigma^j = i\,\varepsilon^{ijk} \sigma^k + \delta^{ij}$$

$$\therefore \quad \gamma^i \gamma^j = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\begin{pmatrix} \sigma^i \sigma^j & 0 \\ 0 & \sigma^i \sigma^j \end{pmatrix} = -\left(i\,\varepsilon^{ijk} \sigma^k + \delta^{ij}\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\rightarrow \quad \gamma^j \gamma^i = -\left(-i\,\varepsilon^{ijk} \sigma^k + \delta^{ij}\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\therefore \quad \sigma^{ij} = \frac{i}{2}(\gamma^i \gamma^j - \gamma^j \gamma^i) = \varepsilon^{ijk} \sigma^k \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$M_1 = \frac{c\,\hbar\, q^*}{(2\pi)^3} \frac{c}{\sqrt{\omega_{k'}\,\omega_k}} A_i(\boldsymbol{k}-\boldsymbol{k}')\, \overline{u}_{\sigma'}(\boldsymbol{k}') \frac{1}{2} \left[k'^{i} + k^{i} + i\,\sigma^{ij} (k'_j - k_j) \right] u_\sigma(\boldsymbol{k})\, e^{-i(\omega_k - \omega_{k'})t}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \int \frac{d^3 k}{(2\pi)^3} \boldsymbol{A}(\boldsymbol{k})\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\rightarrow \qquad \nabla\times\boldsymbol{A}(\boldsymbol{r}) = -i \int \frac{d^3 k}{(2\pi)^3} \boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k})\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}}$$

i.e., the Fourier transform of $\nabla\times\boldsymbol{A}(\boldsymbol{r})$ is $-i\,\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k})$.

Let $\kappa_j = k_j - k'_j$, then (κ_j & A_i are covariant components),

$$i\,A_i(\boldsymbol{k}-\boldsymbol{k}')\,\sigma^{ij}(k'_j - k_j) = -i\,A_i(\boldsymbol{\kappa})\,\sigma^{ij}\,\kappa_j \;= -i\,A_i(\boldsymbol{\kappa})\,\kappa_j\,\varepsilon^{ijk}\,\sigma^k \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= -i\,\boldsymbol{\sigma}\cdot[\boldsymbol{A}(\boldsymbol{\kappa})\times\boldsymbol{\kappa}] \;= -\boldsymbol{\sigma}\cdot\boldsymbol{B}(\boldsymbol{\kappa})$$

Hence,

$$M_1 = \frac{c\,\hbar\, q^*}{(2\pi)^3} \frac{c}{\sqrt{\omega_{k'}\,\omega_k}} \overline{u}_{\sigma'}(\boldsymbol{k}') \frac{1}{2} [-\boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{k}'+\boldsymbol{k}) - \boldsymbol{\sigma}\cdot\boldsymbol{B}]\, u_\sigma(\boldsymbol{k})\, e^{-i(\omega_k-\omega_{k'})t}$$

$$= \frac{1}{(2\pi)^3} \frac{m\,c^2}{\hbar\sqrt{\omega_{k'}\,\omega_k}} \overline{u}_{\sigma'}(\boldsymbol{k}') \left[-\frac{q\,\hbar}{2\,m\,c} \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{k}'+\boldsymbol{k}) - \frac{q\,\hbar}{m\,c} \boldsymbol{S}\cdot\boldsymbol{B} \right] u_\sigma(\boldsymbol{k})\, e^{-i(\omega_k-\omega_{k'})t}$$

where $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the spin (in units of $\hbar$) of the particle.

The term proportional to $\boldsymbol{S}\cdot\boldsymbol{B}$ gives the Zeeman energy.

In terms of the Bohr magneton $\mu_B = \frac{|\,e\,|\,\hbar}{2\,m\,c}$, we can write for an electron

$$\frac{q\,\hbar}{m\,c} \boldsymbol{S}\cdot\boldsymbol{B} = g\,\mu_B\,\boldsymbol{S}\cdot\boldsymbol{B}$$

where the gyromagnetic ratio is $g = 2$.

A check on the sign:

For $q^* > 0$, with particle at rest ($\boldsymbol{k} = \boldsymbol{k}' = 0$) & without spin flip ($\sigma = \sigma'$) , M_1 is negative when $\boldsymbol{\sigma} // \boldsymbol{B}$, as expected.

$\langle \boldsymbol{p}', \sigma' \,|\, H_{\text{int}} \,|\, \boldsymbol{p}, \sigma \rangle_{\text{positron}}$

The evaluation of

$$M = \langle \boldsymbol{p}', \sigma' \,|\, H_{\text{int}} \,|\, \boldsymbol{p}, \sigma \rangle$$

for 2 positron states

$$|\,\boldsymbol{p}, \sigma\rangle = d^+_\sigma(\boldsymbol{k})\,|\,0\rangle \qquad \& \qquad |\,\boldsymbol{p}', \sigma'\rangle = d^+_{\sigma'}(\boldsymbol{k}')\,|\,0\rangle$$

is analogous to the case for electrons. Nonetheless, there're subtleties that must be dealt with carefully.

First, the only surviving term in the product $\overline{\psi}(x)\,\psi(x)$ is:

$$\langle 0 \,|\, d_{\sigma'}(\boldsymbol{k}')\, d_{\sigma_2}(\boldsymbol{k}_2)\, d^+_{\sigma_1}(\boldsymbol{k}_1)\, d^+_\sigma(\boldsymbol{k}) \,|\, 0 \rangle$$

which leads to 2 terms

1. $-\langle 0 \,|\, d_{\sigma'}(\boldsymbol{k}')\, d^+_{\sigma_1}(\boldsymbol{k}_1)\, d_{\sigma_2}(\boldsymbol{k}_2)\, d^+_\sigma(\boldsymbol{k}) \,|\, 0 \rangle = -\delta_{\sigma'\sigma_1}\,\delta(\boldsymbol{k}'-\boldsymbol{k}_1)\,\delta_{\sigma\sigma_2}\,\delta(\boldsymbol{k}-\boldsymbol{k}_2)$
2. $\langle 0 \,|\, d_{\sigma'}(\boldsymbol{k}')\, d^+_\sigma(\boldsymbol{k}) \,|\, 0 \rangle\, \delta_{\sigma_1\sigma_2}\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_2) = \delta_{\sigma'\sigma}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta_{\sigma_1\sigma_2}\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)$

Once again, term 2 leads to self-energies.

After the $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ integration,

$$M_1 = -\frac{c\,\hbar\, q^*}{(2\pi)^3} \int d^3 r \frac{m^*\,c}{\sqrt{\omega_{k'}\,\omega_k}} A_\mu(\boldsymbol{r})\, \overline{v}_\sigma(\boldsymbol{k})\, \gamma^\mu\, v_{\sigma'}(\boldsymbol{k}')\, e^{-i(\omega_k-\omega_{k'})t + i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}$$

$$= -\frac{c\,\hbar\,q^*}{(2\,\pi)^3}\,\frac{m^*\,c}{\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,A_\mu(\boldsymbol{k}-\boldsymbol{k}')\,\overline{v}_\sigma(\boldsymbol{k})\,\gamma^\mu\,v_{\sigma'}(\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

$$= -\frac{1}{(2\,\pi)^3}\,\frac{q\,m\,c^2}{\hbar\,\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,A_\mu(\boldsymbol{k}-\boldsymbol{k}')\,\overline{v}_\sigma(\boldsymbol{k})\,\gamma^\mu\,v_{\sigma'}(\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

Thus, the rules for adapting the electron case are

$$u(\boldsymbol{k}) \to v(\boldsymbol{k}') \qquad\qquad \overline{u}(\boldsymbol{k}') \to \overline{v}(\boldsymbol{k})$$

& an overall − sign for M_1. The switch of $\boldsymbol{k}$, $\boldsymbol{k}'$ can be interpreted as anti-particles traveling backwards in time.

$$\to \qquad M_1 = -\frac{c\,\hbar\,q^*}{(2\,\pi)^3}\,\frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,A_\mu(\boldsymbol{k}-\boldsymbol{k}')\,\overline{v}_\sigma(\boldsymbol{k})\left(k'^{\,\mu}-i\,\sigma^{\mu\nu}\,k'_\nu\right)v_{\sigma'}(\boldsymbol{k}')\;e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

$$= -\frac{c\,\hbar\,q^*}{(2\,\pi)^3}\,\frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,A_\mu(\boldsymbol{k}-\boldsymbol{k}')\,\overline{v}_\sigma(\boldsymbol{k})\left(k^{\mu}+i\,\sigma^{\mu\nu}\,k_\nu\right)v_{\sigma'}(\boldsymbol{k}')\;e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

$$= -\frac{c\,\hbar\,q^*}{(2\,\pi)^3}\,\frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,A_\mu(\boldsymbol{k}-\boldsymbol{k}')\,\overline{v}_\sigma(\boldsymbol{k})\,\frac{1}{2}\left[k'^{\mu}+k^{\mu}\;-i\,\sigma^{\mu\nu}(k'_\nu-k_\nu)\right]v_{\sigma'}(\boldsymbol{k}')\;e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

For the static field,

$$M_1 = -\frac{c\,\hbar\,q^*}{(2\,\pi)^3}\,\frac{c}{\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,\overline{v}_\sigma(\boldsymbol{k})\,\frac{1}{2}\left[\boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{k}'+\boldsymbol{k})+\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]v_{\sigma'}(\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

$$= -\frac{1}{(2\,\pi)^3}\,\frac{m\,c^2}{\hbar\,\sqrt{\omega_{\boldsymbol{k}'}\,\omega_{\boldsymbol{k}}}}\,\overline{v}_\sigma(\boldsymbol{k})\left[\frac{q\,\hbar}{2\,m\,c}\,\boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{k}'+\boldsymbol{k})+\frac{q\,\hbar}{m\,c}\,\boldsymbol{S}\cdot\boldsymbol{B}\right]v_{\sigma'}(\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}$$

We've kept the overall minus sign in deference to the normalization

$$\overline{v}_{\sigma'}(\boldsymbol{k})\,v_\sigma(\boldsymbol{k}) = -\delta_{\sigma'\sigma}$$

Note also that q here is still the charge of the particle. Comparing both M_1's, we see that they obey the rule that the charges of particle & antiparticle have opposite signs.

Thus, in the rest frame, the Zeeman energy of an antiparticle has the opposite sign to that of a particle with the same spin.