Nonrelativistic Limit

In this section, the reduced notations m^*, q^*, ... are suspended since they do not simplify.

From 6.1._DiracEquation.pdf (§ Standard Representation) , we have

$$H = c\begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m c \end{pmatrix}$$

Minimal coupling:

$$p_\mu \;\to\; p_\mu - \frac{q}{c} A_\mu$$

$$\to \qquad H - q\,\varphi = c\begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) \\ \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) & -m c \end{pmatrix}$$

$$H = c\begin{pmatrix} m c + q\,\varphi & \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) \\ \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) & -m c + q\,\varphi \end{pmatrix}$$

In the following, we'll set $\varphi = 0$ & deal omly with

$$H = c\begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) \\ \boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right) & -m c \end{pmatrix}$$

When H operates on plane wave spinors, $\boldsymbol{p}$ can be replace by $\hbar\boldsymbol{k}$ so that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = \boldsymbol{a}^2 \qquad \to \qquad \left[\boldsymbol{\sigma}\cdot\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)\right]^2 = \left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2$$

& the eigenvalues of H is given by

$$\left(m c^2 - E\right)\left(-m c^2 - E\right) - c\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2 = 0$$

$$\to \qquad E = \pm\sqrt{c^2\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2 + \left(m c^2\right)^2}$$

In the nonrelativistic limit,

$$m c^2 >> c\left|\,\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\,\right|$$

$$\to \qquad E = \pm m c^2\left[1 + \frac{\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2}{2\,m^2\,c^2} + \ldots\right]$$

$$\simeq \pm\frac{1}{2\,m}\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2$$

where we've dropped the irrelevant rest energy $m c^2$.

When H operates on a general state,

$$\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right)\right]^2 \neq \left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right)^2$$

due to the differential nature of $\boldsymbol{p}$ (see below).

However, since there is only 1 term, the nonrelativistic limit expression is applicable to a general state if we replace $\left(\hbar\boldsymbol{k}-\frac{q}{c}\boldsymbol{A}\right)^2$ with $\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right)\right]^2$. This gives us the Pauli Hamiltonian

$$H_P = \frac{1}{2m}\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)\right]^2$$

so that

$$H = \begin{pmatrix} H_P & 0 \\ 0 & -H_P \end{pmatrix}$$

Since we're dealing only with spatial vectors, we'll use a_j to denote the j^{th} contravariant component of vector a.

Consider the expression

$$h = [\boldsymbol{\sigma}\cdot(\nabla + \boldsymbol{A})]^2 f \;= \sigma_i\,(\partial_i + A_i)\,\sigma_j\,(\partial_j + A_j) f$$
$$= \sigma_i\,\sigma_j\,(\partial_i + A_i)\,(\partial_j f + A_j f)$$
$$= \sigma_i\,\sigma_j\left[\partial_i \partial_j f + \partial_i (A_j f) + A_i\,\partial_j f + A_i\,A_j f\right]$$

If we apply

$$\sigma_i\,\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k$$

to h, the δ_{ij} term gives

$$h_1 = \nabla^2 f + \nabla\cdot(\boldsymbol{A} f) + \boldsymbol{A}\cdot\nabla f + \boldsymbol{A}^2 f$$
$$= (\nabla + \boldsymbol{A})^2 f$$

while the ε_{ijk} term gives (terms even in i, j vanishes since $\varepsilon_{ijk} = -\varepsilon_{jik}$)

$$h_2 = i\,\varepsilon_{ijk}\,\sigma_k\left[\partial_i (A_j f) + A_i\,\partial_j f\right]$$
$$= \left(i\,\varepsilon_{ijk}\,\sigma_k\,\partial_i A_j\right) f$$
$$= i\,\boldsymbol{\sigma}\cdot(\nabla\times\boldsymbol{A}) f$$

Since f is arbitrary, we have

$$[\boldsymbol{\sigma}\cdot(\nabla + \boldsymbol{A})]^2 = (\nabla + \boldsymbol{A})^2 + i\,\boldsymbol{\sigma}\cdot(\nabla\times\boldsymbol{A})$$

As stated previously, the $\nabla\times\boldsymbol{A}$ term is due to the presence of ∇ on the left hand side.

Thus,

$$H_P\,\psi = \frac{1}{2m}\left[\boldsymbol{\sigma}\cdot\left(-i\hbar\nabla - \frac{q}{c}\boldsymbol{A}\right)\right]^2 \psi$$
$$= \frac{1}{2m}\left[\left(-i\hbar\nabla - \frac{q}{c}\boldsymbol{A}\right)^2 + (-i\hbar)^2\, i\,\boldsymbol{\sigma}\cdot\left(\nabla\times\frac{q}{i\hbar c}\boldsymbol{A}\right)\right]\psi$$
$$= \left[\frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 - \frac{q\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]\psi$$

ψ is arbitray

$$\rightarrow \qquad H_P = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 - \frac{q\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

$$H_P = \frac{1}{2m}\boldsymbol{p}^2 - \frac{q}{2mc}(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) + \frac{q^2}{2mc^2}\boldsymbol{A}^2 - \frac{q\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

$$\rightarrow \qquad H_{\text{int}} = -\frac{q}{2mc}(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) - \frac{q\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

For electrons with charge $q = -e$ & spin $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$,

$$-\frac{q\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B} = \frac{e\hbar}{mc}\,\boldsymbol{S}\cdot\boldsymbol{B} \;= g\,\mu_B\,\boldsymbol{S}\cdot\boldsymbol{B} \qquad\qquad \left(\mu_B = \frac{e\hbar}{2mc}, \quad g = 2\right)$$