6.5. Weyl Field

Massless Dirac theory (no rest frame) :

$$\mathcal{L} = i\hbar c\,\overline{\psi}\,\gamma^{\mu}\,\partial_{\mu}\psi$$

In 6.1._DiracEquation.pdf (§ Noether Currents), we've shown that $\mathcal{L}$ is invariant under the chiral transformation:

$$\psi(x) \;\to\; e^{i f \gamma_5}\,\psi(x) \qquad f = \text{real constant}$$

$$\to \qquad \overline{\psi}(x) \;\to\; \overline{\psi}(x)\,e^{-i\epsilon\gamma_5}$$

with a Noether (pseudo-) current

$$j^{\mu} = \hbar c\,\overline{\psi}\,\gamma^{\mu}\,\gamma_5\,\psi$$

Hence, $\mathcal{H}$ & $H = \int d^3 r\,\mathcal{H}$ are also invariant.

In the standard representation,

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

&

$$H = c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} = c\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\gamma_5$$

$$\to \qquad [\gamma_5\,,\; H] = 0$$

Chiral States

Eigen-problem:

$$\gamma_5\,\phi = \lambda\,\phi$$

$$\to \qquad \begin{pmatrix} -\lambda & I \\ I & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\therefore \qquad \lambda^2 - 1 = 0 \qquad \to \qquad \lambda = \pm 1 \equiv \lambda_{\pm}$$

$$\& \qquad \phi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \phi_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[\gamma_5\,,\; H] = 0$$

$\to$ H is also block diagonal in the basis $\{\phi_{\pm}\}$, i.e.,

$$H = \frac{c}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = c\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}$$

In this basis, the solutions to the Dirac eq.

$$i\hbar\partial_t\,\psi = H\,\psi$$

are

$$\psi_+ = \begin{pmatrix} \varphi_+ \\ 0 \end{pmatrix} \qquad \psi_- = \begin{pmatrix} 0 \\ \varphi_- \end{pmatrix}$$

with

$$i\hbar\partial_t\varphi_{\pm} = \pm c\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\varphi_{\pm} \;\; = \mp\, i\hbar c\,\boldsymbol{\sigma}\cdot\nabla\varphi_{\pm}$$

where the subscript + (–) denotes the left- (right-) handed chirality.

Energy-Chiral Eigenstates

Let us solve the eigen-problem

$$H(\boldsymbol{k})\,\psi(\boldsymbol{k}) = \hbar c \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix} \psi = E\,\psi$$

Note that

$$H\,\psi = c \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} \psi(\boldsymbol{r})$$

reduces to our problem only if the spinor $\psi(\boldsymbol{r}) \propto e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}}$.

Consider

$$f = a + b\,\boldsymbol{\sigma}\cdot\boldsymbol{k}$$

where a, b are constants.

$$(\boldsymbol{\sigma}\cdot\boldsymbol{k})^2 = \boldsymbol{k}^2$$

$$\rightarrow \qquad \boldsymbol{\sigma}\cdot\boldsymbol{k}\,f = a\,\boldsymbol{\sigma}\cdot\boldsymbol{k} + b\,\boldsymbol{k}^2 = \frac{b\,\boldsymbol{k}^2}{a}\left(\frac{a^2}{b\,\boldsymbol{k}^2}\,\boldsymbol{\sigma}\cdot\boldsymbol{k} + a\right)$$

Let $b = \pm \dfrac{a}{|\,\boldsymbol{k}\,|}$ $\qquad \rightarrow \qquad$ $\dfrac{a^2}{b\,\boldsymbol{k}^2} = b$

Hence,

$$\boldsymbol{\sigma}\cdot\boldsymbol{k}\,f_\pm = \pm\,|\,\boldsymbol{k}\,|\,f_\pm$$

where $\qquad f_\pm = a_\pm\left(1 \pm \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{|\,\boldsymbol{k}\,|}\right) = a_\pm\left(1 \pm \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)$

$\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\,\boldsymbol{p}\,|} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$ is known as the helicity operator.

The eigenvectors of $H(\boldsymbol{k})$ are therefore of the form

$$\frac{1}{\sqrt{2}}\,f_s\begin{pmatrix} 1 \\ s' \end{pmatrix} = f_s\,\phi_{s'} \qquad\qquad s,\ s' = \pm$$

which has 4 combinations.

Reminder: $\phi_\pm$ are the chiral (γ_5) eigenstates.

Thus,

$$\hbar c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix} f_+\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \hbar c\,\big|\,\boldsymbol{k}\,\big|\,f_+\begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$$

$$\rightarrow \qquad H(\boldsymbol{k})\,(f_+\,\phi_\pm) = \pm\,\hbar c\,|\,\boldsymbol{k}\,|\,(f_+\,\phi_\pm)$$

$$\hbar c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix} f_-\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = -\hbar c\,\big|\,\boldsymbol{k}\,\big|\,f_-\begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$$

$$\rightarrow \qquad H(\boldsymbol{k})\,(f_-\,\phi_\pm) = \mp\,\hbar c\,|\,\boldsymbol{k}\,|\,(f_-\,\phi_\pm)$$

Since $E > 0$, we can use only

$$f_+\,\phi_+ = a_+\left(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_+ \qquad \text{or} \qquad f_-\,\phi_- = a_-\left(1 - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_-$$

When $\psi(\boldsymbol{r}) \propto e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}}$, we have

$$H(-\boldsymbol{k})\,\psi = \hbar c\begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & 0 \end{pmatrix}\psi = E\,\psi$$

It's easy to see that

$$H(-\boldsymbol{k})\,(f_+\,\phi_\pm) = \mp\,\hbar c\,|\,\boldsymbol{k}\,|\,(f_+\,\phi_\pm)$$

$$H(-\boldsymbol{k})\ (f_-\ \phi_\pm) = \pm\ \hbar c\ |\ \boldsymbol{k}\ |\ (f_-\ \phi_\pm)$$

Rule $E < 0$ means we can use only

$$f_+\ \phi_+ = a_+\left(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_+ \qquad \text{or} \qquad f_-\ \phi_- = a_-\left(1 - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_-$$

The u spinors are operated by $H(\boldsymbol{k})$ so we have

$$u_R(\boldsymbol{k}) = f_+\ \phi_+ = a_+\left(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_+ \qquad u_L(\boldsymbol{k}) = f_-\ \phi_- = a_-\left(1 - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_-$$

The v spinors are operated by $H(-\boldsymbol{k})$ so we have

$$\to \qquad v_R(\boldsymbol{k}) = f_+\ \phi_+ = a_+\left(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_+ \qquad v_L(\boldsymbol{k}) = f_-\ \phi_- = a_-\left(1 - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\right)\phi_-$$

where we've switched to R & L to denote handedness.

$$\boldsymbol{\sigma}\cdot\boldsymbol{k}\, f_\pm = \pm\ |\ \boldsymbol{k}\ |\ f_\pm$$

$$\to \qquad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ u_R(\boldsymbol{k}) = u_R(\boldsymbol{k}) \qquad -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ v_R(\boldsymbol{k}) = -v_R(\boldsymbol{k})$$

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ u_L(\boldsymbol{k}) = -u_L(\boldsymbol{k}) \qquad -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ v_L(\boldsymbol{k}) = v_L(\boldsymbol{k})$$

These expressions can be written in terms of operator $\boldsymbol{p}$ if we define

$$u_R(\boldsymbol{p}) = u_R(\boldsymbol{k})\ e^{-i\,k\cdot x} \qquad \& \qquad v_R(\boldsymbol{p}) = v_R(\boldsymbol{k})\ e^{i\,k\cdot x}$$

Then we have

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\ u_R(\boldsymbol{p}) = u_R(\boldsymbol{p}) \qquad \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\ v_R(\boldsymbol{p}) = -v_R(\boldsymbol{p})$$

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\ u_L(\boldsymbol{p}) = -u_L(\boldsymbol{p}) \qquad \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\ v_L(\boldsymbol{p}) = v_L(\boldsymbol{p})$$

Thus, the helicity of u_R & v_L is 1, that of u_L & v_R is -1.

Also, we can write

$$u_R(\boldsymbol{p}) = f_+\ \phi_+\ e^{-i\,k\cdot x} = a_+\left(1 + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)\phi_+\ e^{-i\,k\cdot x}$$

$$u_L(\boldsymbol{p}) = f_-\ \phi_-\ e^{-i\,k\cdot x} = a_-\left(1 - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)\phi_-\ e^{-i\,k\cdot x}$$

$$v_R(\boldsymbol{p}) = f_+\ \phi_+\ e^{i\,k\cdot x} = a_+\left(1 - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)\phi_+\ e^{i\,k\cdot x}$$

$$v_L(\boldsymbol{p}) = f_-\ \phi_-\ e^{i\,k\cdot x} = a_-\left(1 + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)\phi_-\ e^{i\,k\cdot x}$$

Values of $a_\pm$ depend on the normalization we choose. Since the covariant normalization leads to a plane wave expansion $\psi(x) \propto \sqrt{m} = 0$ (see § 6.3), our only choice is the hermitian normalization:

$$w_\chi^+(\boldsymbol{k})\ w'_{\chi'}(\boldsymbol{k}) = \delta_{\chi\chi'}\ \delta_{w\,w'} \qquad \text{where} \quad w = u,\ v \ \ \& \ \ \chi = R,\ L$$

Plane Wave Expansion

Plane wave expansion:

$$\psi_R\,(x) = \int \frac{d^3 k}{(2\,\pi)^{3/2}}\ \alpha_{\boldsymbol{k}}\left[\,c_R(\boldsymbol{k})\ u_R(\boldsymbol{k})\ e^{-i\,k\cdot x} + d_R^+(\boldsymbol{k})\ v_R(\boldsymbol{k})\ e^{i\,k\cdot x}\,\right]$$

$$\psi_L\,(x) = \int \frac{d^3 k}{(2\,\pi)^{3/2}}\ \beta_{\boldsymbol{k}}\left[\,c_L(\boldsymbol{k})\ u_L(\boldsymbol{k})\ e^{-i\,k\cdot x} + d_L^+(\boldsymbol{k})\ v_L(\boldsymbol{k})\ e^{i\,k\cdot x}\,\right]$$

where $\alpha_{\boldsymbol{k}}$, $\beta_{\boldsymbol{k}}$ are normalization constants to be determined by our desire to write

$$H = \int d^3 r\,\mathcal{H}$$

$$= \sum_{\chi = R,L} \int d^3 k\ \hbar c\ \left|\ \boldsymbol{k}\ \right|\ \left[c_\chi^+(\boldsymbol{k})\ c_\chi(\boldsymbol{k})\ - d_\chi(\boldsymbol{k})\ d_\chi^+(\boldsymbol{k})\right]$$

with the quantization rules

$$\left\{c_\chi(\boldsymbol{k})\,,\ c_{\chi'}^+(\boldsymbol{k}')\right\} = \delta_{\chi\chi'}\ \delta(\boldsymbol{k} - \boldsymbol{k}')$$

$$\left\{d_\chi(\boldsymbol{k})\,,\ d_{\chi'}^+(\boldsymbol{k}')\right\} = \delta_{\chi\chi'}\ \delta(\boldsymbol{k} - \boldsymbol{k}')$$

& $\{c_\chi(\boldsymbol{k}),\ c_{\chi'}(\boldsymbol{k}')\} = \{c_\chi^+(\boldsymbol{k}),\ c_{\chi'}^+(\boldsymbol{k}')\} = 0$

$\{d_\chi(\boldsymbol{k}),\ d_{\chi'}(\boldsymbol{k}')\} = \{d_\chi^+(\boldsymbol{k}),\ d_{\chi'}^+(\boldsymbol{k}')\} = 0$

$\{c_\chi(\boldsymbol{k}),\ d_{\chi'}(\boldsymbol{k}')\} = \{c_\chi(\boldsymbol{k}),\ d_{\chi'}^+(\boldsymbol{k}')\} = 0$

$\{c_\chi^+(\boldsymbol{k}),\ d_{\chi'}(\boldsymbol{k}')\} = \{c_\chi^+(\boldsymbol{k}),\ d_{\chi'}^+(\boldsymbol{k}')\} = 0$

Hamiltonian

Setting $m = 0$ to the results in 6.3._CanonicalQuantization.pdf , we have

$$\mathcal{L} = i\hbar c\,\overline{\psi}\,\gamma^\mu\,\partial_\mu\psi$$

$$\rightarrow \quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\overline{\psi}\,\gamma^0 = i\hbar\,\psi^+$$

$$\overline{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\overline{\psi}}} = 0$$

$$\therefore \quad \mathcal{H} = -i\,c\,\hbar\,\overline{\psi}\,\boldsymbol{\gamma}\cdot\nabla\psi$$

$$= -i\,c\,\hbar\,\psi^+\,\boldsymbol{\alpha}\cdot\nabla\psi$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\gamma}^0\,\boldsymbol{\gamma}$$

In the standard representation,

$$\boldsymbol{\alpha} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} = \boldsymbol{\sigma}\,\gamma_5$$

$$\rightarrow \quad \mathcal{H} = -i\,c\,\hbar\,\psi^+\,\boldsymbol{\sigma}\,\gamma_5\cdot\nabla\psi$$

$$= -i\,c\,\hbar\,\psi^+\,\eta\,\psi$$

where

$$\eta = -i\,\boldsymbol{\sigma}\,\gamma_5\cdot\nabla$$

$$\rightarrow \quad \eta\,u_R(\boldsymbol{k})\,e^{-i\,k\cdot x} = -i\,\boldsymbol{\sigma}\gamma_5\,u_R(\boldsymbol{k})\cdot\nabla e^{-i\,k\cdot x}$$

$$= \boldsymbol{\sigma}\cdot\boldsymbol{k}\,u_R(\boldsymbol{k})\,e^{-i\,k\cdot x} = \boldsymbol{k}\,u_R(\boldsymbol{k})\,e^{-i\,k\cdot x}$$

$$\eta\,u_L(\boldsymbol{k})\,e^{-i\,k\cdot x} = -\boldsymbol{\sigma}\cdot\boldsymbol{k}\,u_L(\boldsymbol{k})\,e^{-i\,k\cdot x} = \boldsymbol{k}\,u_L(\boldsymbol{k})\,e^{-i\,k\cdot x}$$

$$\eta\,v_R(\boldsymbol{k})\,e^{i\,k\cdot x} = -\boldsymbol{\sigma}\cdot\boldsymbol{k}\,v_R(\boldsymbol{k})\,e^{i\,k\cdot x} = -\boldsymbol{k}\,v_R(\boldsymbol{k})\,e^{i\,k\cdot x}$$

$$\eta\,v_L(\boldsymbol{k})\,e^{i\,k\cdot x} = \boldsymbol{\sigma}\cdot\boldsymbol{k}\,v_L(\boldsymbol{k})\,e^{i\,k\cdot x} = -\boldsymbol{k}\,v_L(\boldsymbol{k})\,e^{i\,k\cdot x}$$

$$\therefore \quad \eta\,\psi_\chi(x) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\,\boldsymbol{k}\left[c_\chi(\boldsymbol{k})\,u_\chi(\boldsymbol{k})\,e^{-i\,k\cdot x} - d_\chi^+(\boldsymbol{k})\,v_\chi(\boldsymbol{k})\,e^{i\,k\cdot x}\right]$$

$$\psi_\chi^+(x) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\alpha_{\boldsymbol{k}}\left[c_\chi^+(\boldsymbol{k})\,u_\chi^+(\boldsymbol{k})\,e^{i\,k\cdot x} + d_\chi(\boldsymbol{k})\,v_\chi^+(\boldsymbol{k})\,e^{-i\,k\cdot x}\right]$$

H is now easily evaluated using the orthnormality of u & v.
Thus,

$$\alpha_{\boldsymbol{k}}^2 = \beta_{\boldsymbol{k}}^2$$

$$\& \quad H = \sum_{\chi=R,L}\int d^3k\,\hbar\,|\boldsymbol{k}|\,\alpha_{\boldsymbol{k}}^2\left[c_\chi^+(\boldsymbol{k})\,c_\chi(\boldsymbol{k}) - d_\chi(\boldsymbol{k})\,d_\chi^+(\boldsymbol{k})\right]$$

$$\therefore \quad \alpha_{\boldsymbol{k}} = \beta_{\boldsymbol{k}} = \sqrt{c}$$

$$\psi_\chi(x) = \sqrt{c} \int \frac{d^3 k}{(2\pi)^{3/2}} \left[c_\chi(\boldsymbol{k})\, u_\chi(\boldsymbol{k})\, e^{-i k\cdot x} + d_\chi^+(\boldsymbol{k})\, v_\chi(\boldsymbol{k})\, e^{i k\cdot x} \right]$$

$$\psi_\chi^+(x) = \sqrt{c} \int \frac{d^3 k}{(2\pi)^{3/2}} \left[c_\chi^+(\boldsymbol{k})\, u_\chi^+(\boldsymbol{k})\, e^{i k\cdot x} + d_\chi(\boldsymbol{k})\, v_\chi^+(\boldsymbol{k})\, e^{-i k\cdot x} \right]$$

→ $$\left\{ \psi_\chi(t, \boldsymbol{r}),\ \psi_{\chi'}^+(t, \boldsymbol{r}') \right\} = c\, \delta_{\chi\chi'}\, \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\left\{ \psi_\chi(t, \boldsymbol{r}),\ \psi_{\chi'}(t, \boldsymbol{r}') \right\} = \left\{ \psi_\chi^+(t, \boldsymbol{r}),\ \psi_{\chi'}^+(t, \boldsymbol{r}') \right\} = 0$$

Chiral / Weyl Representation

The chiral, or Weyl, representation is characterized by having γ_5 diagonal, i.e., switching to the basis $\phi_\pm$.

→ $$\gamma^0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boldsymbol{\gamma} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\gamma_5 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

& $$H = \frac{1}{2} c \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = c \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}$$

$$= c\, \boldsymbol{\sigma}\cdot\boldsymbol{p}\, \gamma_5$$