## 6.6. Dirac Electrons in Magnetic Field

In the presence of a static magnetic field, the Dirac Hamiltonian becomes (see §6.1)

$$H = c\begin{pmatrix} m c & \boldsymbol{\sigma}\cdot\boldsymbol{P} \\ \boldsymbol{\sigma}\cdot\boldsymbol{P} & -m c \end{pmatrix}$$

where

$$\boldsymbol{P} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A}$$

For a uniform $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, the simplest solutions to

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = (0,\ 0,\ B)$$

are those with $A_z = 0$, e.g.,

$$\boldsymbol{A} = (0,\ x B,\ 0) \quad \text{or} \qquad \boldsymbol{A} = \frac{1}{2}(-y B,\ x B,\ 0)$$

If the particles are also confined to the xy-plane, we have

$$\boldsymbol{\sigma}\cdot\boldsymbol{P} = \begin{pmatrix} 0 & P_- \\ P_+ & 0 \end{pmatrix} \equiv Q$$

where

$$P_\pm = P_x \pm i P_y$$

$$\begin{aligned}\left[P_x,\ P_y\right] &= \left[p_x - \frac{q}{c}A_x,\ p_y - \frac{q}{c}A_y\right] \\ &= \left[p_x,\ -\frac{q}{c}A_y\right] + \left[-\frac{q}{c}A_x,\ p_y\right] \\ &= i\hbar\frac{q}{c}\left(\partial_x A_y - \partial_y A_x\right) \\ &= i\hbar\frac{q}{c}B\end{aligned}$$

$\rightarrow$

$$\begin{aligned}\left[P_+,\ P_-\right] &= \left[P_x,\ -i P_y\right] + \left[i P_y,\ P_x\right] \\ &= -2i\left[P_x,\ P_y\right] \\ &= 2\hbar\frac{q}{c}B\end{aligned}$$

Let

$$a = \sqrt{\frac{c}{2\hbar q B}}\,P_+ \qquad\qquad a^+ = \sqrt{\frac{c}{2\hbar q B}}\,P_-$$

$\rightarrow$

$$[a,\ a^+] = 1$$

&

$$Q = \sqrt{\frac{2\hbar q B}{c}}\begin{pmatrix} 0 & a^+ \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix}$$

where

$$A = \sqrt{\frac{2\hbar q B}{c}}\,a$$

$$\therefore \qquad Q^2 = \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix} = \begin{pmatrix} A^+ A & 0 \\ 0 & A A^+ \end{pmatrix}$$

$$= \frac{2 \hbar q B}{c} \begin{pmatrix} a^+ a & 0 \\ 0 & a a^+ \end{pmatrix}$$

The associated Fock states

$$| \, N \rangle = \frac{1}{\sqrt{N}} (a^+)^N \, \left| \, 0 \right\rangle$$

are called the $N^{\text{th}}$ Landau levels:

Also

$$H = c \begin{pmatrix} m c & Q \\ Q & -m c \end{pmatrix}$$

$$\to \qquad H^2 = c^2 \begin{pmatrix} m c & Q \\ Q & -m c \end{pmatrix} \begin{pmatrix} m c & Q \\ Q & -m c \end{pmatrix}$$

$$= c^2 \begin{pmatrix} (m c)^2 + Q^2 & 0 \\ 0 & (m c)^2 + Q^2 \end{pmatrix}$$

$$\therefore \qquad H = c \begin{pmatrix} \sqrt{(m c)^2 + Q^2} & 0 \\ 0 & -\sqrt{(m c)^2 + Q^2} \end{pmatrix}$$

Note that both ( ± ) roots of square root $(m c)^2 + Q^2$ are needed to ensure all eigenvalues of $H$ are covered.

In the non-relativistic limit,

$$c \sqrt{(m c)^2 + Q^2} \approx m c^2 + \frac{Q^2}{2 m}$$

$$\to \qquad H \approx \frac{Q^2}{2 m} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

while

$$\frac{Q^2}{2 m} = \hbar \frac{q B}{m c} \begin{pmatrix} a^+ a & 0 \\ 0 & a a^+ \end{pmatrix} = \hbar \omega_c \begin{pmatrix} a^+ a & 0 \\ 0 & a a^+ \end{pmatrix}$$

where

$$\omega_c = \frac{q B}{m c}$$

is the cyclotron frequency.

Setting

$$k_Q = \sqrt{\frac{2 m \omega_c}{\hbar}} = \sqrt{\frac{2 q B}{\hbar c}}$$

we have

$$Q^2 = (\hbar k_Q)^2 \begin{pmatrix} a^+ a & 0 \\ 0 & a a^+ \end{pmatrix}$$

Thus

$$\left\langle N \,\middle|\, Q^2 \,\middle|\, N \right\rangle = (\hbar k_Q)^2 \begin{pmatrix} N & 0 \\ 0 & N + 1 \end{pmatrix}$$

$$\left\langle N \left| \sqrt{(m c)^2 + Q^2} \right| N \right\rangle = \frac{1}{c} \begin{pmatrix} \mathcal{E}(N) & 0 \\ 0 & \mathcal{E}(N+1) \end{pmatrix}$$

where

$$\mathcal{E}(N) = c \sqrt{(m c)^2 + N (\hbar k_Q)^2}$$

$$\therefore \qquad \langle N \mid H \mid N \rangle = \begin{pmatrix} \mathcal{E}(N) & 0 & 0 & 0 \\ 0 & \mathcal{E}(N+1) & 0 & 0 \\ 0 & 0 & -\mathcal{E}(N) & 0 \\ 0 & 0 & 0 & -\mathcal{E}(N+1) \end{pmatrix}$$

The eigen-spinors are simply

$$u_\uparrow^{\mathcal{E}(N)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_\downarrow^{\mathcal{E}(N+1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_\uparrow^{-\mathcal{E}(N)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_\downarrow^{-\mathcal{E}(N+1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, associated with a given Landau level $| N \rangle$, we have 2 energy eigenstates of positive energies: $\mathcal{E}(N)$ for spin ↑ particles & $\mathcal{E}(N+1)$ for spin ↓ particles.

Conversely, a given energy level $\mathcal{E}(N)$, except for $N = 0$, can belong to either a spin ↑ particle in Landau level $| N \rangle$ or a spin ↓ particle in $| N - 1 \rangle$.

For the vacuum $| 0 \rangle$, we have a spin ↑ particle with energy $\mathcal{E}(0) = m c^2$, & a spin ↓ particle with energy $\mathcal{E}(1) = c \sqrt{(m c)^2 + (\hbar k_Q)^2}$. (see Ezawa, Fig.6.2)

Comparing the Hamiltonian

$$H_Q = c^2 Q^2 = c^2 (\boldsymbol{\sigma} \cdot \boldsymbol{P})^2$$

with the Pauli Hamiltonian (see 6.4.a._NonrelativisticLimit.pdf )

$$H_P = \frac{1}{2m} (\boldsymbol{\sigma} \cdot \boldsymbol{P})^2 = \frac{1}{2m} \left( \boldsymbol{p} - \frac{q}{c} \boldsymbol{A} \right)^2 - \frac{q \hbar}{2 m c} \boldsymbol{\sigma} \cdot \boldsymbol{B}$$

$$= \frac{1}{2m} \left( \boldsymbol{p} - \frac{q}{c} \boldsymbol{A} \right)^2 - \frac{1}{2} \hbar \omega_c \boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}} \qquad \left( \omega_c = \frac{q B}{m c} \right)$$

we see that $H_Q$ is just a Pauli Hamiltonian with an effective mass $m_Q = \frac{1}{2 c^2}$.

$$\rightarrow \qquad H_Q = c^2 \left( \boldsymbol{p} - \frac{q}{c} \boldsymbol{A} \right)^2 - q \hbar c \boldsymbol{\sigma} \cdot \boldsymbol{B}$$

Note that the Zeeman energies of $H_P$ are $\pm \frac{1}{2} \hbar \omega_c$,

while the cyclotron energy is $\hbar \omega_c$.

Thus, the spectrum $\pm \mathcal{E}(N)$ of the relativistic Dirac Hamiltonian $H$ are constructed from the spectrum

$$E(N) = N (\hbar c k_Q)^2 = 2 q \hbar c B N$$

of the non-relativistic Hamiltonian $H_Q$.

For $H_Q$, the Zeeman effect associated with the $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ term is called intrinsic since $\boldsymbol{\sigma}$ is a built-in property of the Dirac Hamiltonian.

The intrinsic Zeeman energies are $E_Z = \pm q \hbar c B$.
The cyclotron energy in $H_Q$ is

$$\hbar \omega_c^Q = \hbar \frac{q B}{m_Q c} = 2 q \hbar c B$$

Caution: These “energies” have dimensions of $(\text{energy})^2$ since we’ve set $m_Q\, c^2 = \frac{1}{2}$ dimensionless.

On the other hand, associated with $k_Q = \sqrt{\frac{2\, q\, B}{\hbar\, c}}$ is a frequency

$$\omega_Q = k_Q\, c$$

so that $\mathcal{E}(N) = \sqrt{\left(m\, c^2\right)^2 + N\, (\hbar\, \omega_Q)^2}$ .

Note that $\hbar\, \omega_Q = \hbar\, k_Q\, c = \sqrt{2\, q\, \hbar\, c\, B} = \sqrt{\hbar\, \omega_c^Q}$ .