6.6.a. SUSY Quantum Mechanics

Introduction to SUSY QM

Ref: F.Cooper et al, “Supersymmetry in Quantum Mechanics”, Chap.3.

Consider the 1-D Hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$

If V is bounded below, we can set the ground state energy $E_0 = 0$ so that the Schrodinger eq. for the ground state ψ_0 becomes

$$H\,\psi_0 = \left(-\frac{\hbar^2}{2m}\frac{d^2}{d x^2} + V \right) \psi_0 = 0$$

$$\rightarrow \qquad V = \frac{\hbar^2}{2m}\frac{d^2\psi_0}{\psi_0\, d x^2}$$

i.e., if ψ_0 is known, one can contruct a V that gives rise to it.

We wish to write

$$H = A^+ A$$

where

$$A = i\frac{p}{\sqrt{2m}} + W(x) \qquad\qquad A^+ = -i\frac{p}{\sqrt{2m}} + W(x)$$

Thus

$$H = \left(-i\frac{p}{\sqrt{2m}} + W \right)\left(i\frac{p}{\sqrt{2m}} + W \right)$$

$$= \frac{p^2}{2m} + W^2 - \frac{i}{\sqrt{2m}}(p\,W - W\,p)$$

$$= \frac{p^2}{2m} + W^2 - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}$$

$$\rightarrow \qquad V = W^2 - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}$$

The solution to this is

$$W = -\frac{\hbar}{\sqrt{2m}}\frac{d\psi_0}{\psi_0\, dx}$$

Proof:

$$W^2 = \frac{\hbar^2}{2m}\left(\frac{d\psi_0}{\psi_0\, dx} \right)^2$$

$$-\frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} = \frac{\hbar^2}{2m}\left(\frac{d^2\psi_0}{\psi_0\, dx^2} - \left(\frac{d\psi_0}{\psi_0\, dx} \right)^2 \right)$$

$$\rightarrow \qquad W^2 - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} = \frac{\hbar^2}{2m}\frac{d^2\psi_0}{\psi_0\, dx^2} = V$$

By the switching $A \leftrightarrow A^+$, we obtain the supersymmetric partners of H & V,

$$H' = A\,A^+ = \frac{p^2}{2m} + V'(x)$$

$$V' = W^2 + \frac{i}{\sqrt{2m}}(p W - W p)$$
$$= W^2 + \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}$$
$$= 2 W^2 - V$$

$\to \qquad V + V' = 2 W^2$

Let ψ_n & ψ'_n be the n^{th} eigenstates, with eigenvalues E_n & E'_n, of H & H', respectively.

$$H \psi_n = A^+ A \ \psi_n = E_n \psi_n$$

$A \times$ both sides gives

$$A A^+ A \ \psi_n = H' (A \ \psi_n) = E_n (A \psi_n)$$

$\to \qquad A \psi_n$ is an eigenstate of H' with eigenvalue E_n.

However, for $n = 0$,

$$E_0 = 0 \qquad \to \qquad A \psi_0 = 0$$

so that the eigenstates of H' start with $A \psi_1$, i.e.,

$$\psi'_0 \propto A \psi_1 \qquad \& \qquad E'_0 = E_1$$

$\to \qquad \psi'_n \propto A \psi_{n+1} \qquad \& \qquad E'_n = E_{n+1} \qquad \forall \ n = 0, 1, 2, \ldots$

Conversely,

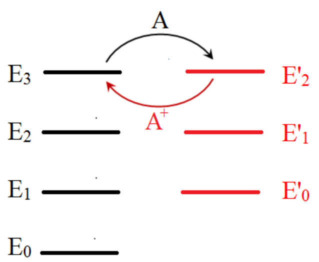
$$A^+ H' \psi'_n = A^+ A A^+ \psi'_n = H \left(A^+ \psi'_n \right) = E'_n \left(A^+ \psi'_n \right)$$

$\to \qquad A^+ \psi'_n$ is an eigenstate of H with eigenvalue E'_n.

Since we already established $E'_n = E_{n+1}$

$\therefore \qquad \psi_{n+1} \propto A^+ \psi'_n \qquad \forall \ n = 0, 1, 2, \ldots$

which means ψ_0 has no super-partner.



Thus, the solutions to a rather difficult eigen-problem

$$H' \psi' = E' \psi'$$

can be obtained with ease if its superpartner problem

$$H \psi = E \psi$$

are easily solvable. In which case,

$$\psi'_n \propto A \psi_{n+1} \qquad \& \qquad E'_n = E_{n+1}$$

See Cooper for examples of such applications.

We now normalize the eigenstates. Let

$$\psi_{n+1} = c A^+ \psi'_n \to \qquad \psi^+_{n+1} = c^* \psi'^+_n A$$

$\to \qquad \langle \psi_{n+1} \mid \psi_{n+1} \rangle = c^* c \left\langle \psi'_n \mid A A^+ \mid \psi'_n \right\rangle = c^* c E'_n \langle \psi'_n \mid \psi'_n \rangle$

$$= c^* c E_{n+1} \langle \psi'_n \mid \psi'_n \rangle$$

$\therefore \qquad \psi_{n+1} = \frac{1}{\sqrt{E_{n+1}}} A^+ \psi'_n = \frac{1}{\sqrt{E'_n}} A^+ \psi'_n$

are all normalized if the ψ'_n are.

Thus,

$$A\psi_{n+1} = \frac{1}{\sqrt{E_{n+1}}} AA^+ \psi'_n$$

Setting $\psi'_n = c' A \psi_{n+1}$

$$\rightarrow \qquad \frac{1}{c'} \psi'_n = \frac{1}{\sqrt{E_{n+1}}} E'_n \psi'_n = \sqrt{E_{n+1}} \; \psi'_n$$

$$\therefore \qquad c' = \frac{1}{\sqrt{E_{n+1}}} = \frac{1}{\sqrt{E'_n}}$$

& the normalized superstates are

$$\psi'_n = \frac{1}{\sqrt{E'_n}} A \psi_{n+1} = \frac{1}{\sqrt{E_{n+1}}} A \psi_{n+1}$$

SUSY Hamiltonian

The two spectra can be combined into a SUSY Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H & 0 \\ 0 & H' \end{pmatrix} = \begin{pmatrix} A^+ A & 0 \\ 0 & A A^+ \end{pmatrix}$$

with 2 kinds of eigenstates :

$$\Psi_n = \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} \qquad \& \qquad \Psi'_n = \begin{pmatrix} 0 \\ \psi'_n \end{pmatrix}$$

and degenerate energies

$$E'_n = E_{n+1}$$

The cross over of states can be effected by the operators

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \qquad \& \qquad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$$

For example,

$$\psi_{n+1} = \frac{1}{\sqrt{E'_n}} A^+ \psi'_n$$

now becomes

$$\Psi_{n+1} = \frac{1}{\sqrt{E'_n}} Q^+ \Psi'_n$$

the validity of which is easily checked.

Note that

$$Q^2 = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0 \qquad \& \qquad Q^{+2} = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = 0$$

$$\rightarrow \qquad \{Q, Q\} = 0 \qquad \& \qquad \{Q^+, Q^+\} = 0$$

Thus, Q or Q^+ anti-commutes with itself, so we may call them Fermion operators.

Calling the space spanned by Ψ_n (Ψ'_n) the bosonic (fermionic) sector, Q & Q^+ thus performs the transmutation between bosons & fermions, hence the name SUper-SYmmetry (SUSY).

Also,

$$Q Q^+ = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A A^+ \end{pmatrix}$$

$$Q^+ Q = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} A^+ A & 0 \\ 0 & 0 \end{pmatrix}$$

→ $\{Q, Q^+\} = \mathcal{H}$

& $$Q \mathcal{H} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} A^+ A & 0 \\ 0 & A A^+ \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A A^+ A & 0 \end{pmatrix}$$

$$\mathcal{H} Q = \begin{pmatrix} A^+ A & 0 \\ 0 & A A^+ \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A A^+ A & 0 \end{pmatrix}$$

→ $[Q, \mathcal{H}] = 0$

Using $\{Q, Q^+, \mathcal{H}, I\}$ as basis, together with the anti- & commutation rules,

$\{Q, Q\} = 0$ $\{Q^+, Q^+\} = 0$ $\{Q, Q^+\} = \mathcal{H}$

$[Q, \mathcal{H}] = 0$

one can construct a closed algebra classified as sl(1/1), where sl stands for superlinear & 1/1 means that the irreducible representation that defines the algebra are block matrices of dimensions $(1+1)\times(1+1)$.

Q^2

The general Pauli Hamiltonian:

$$H_Q = c^2 Q^2 = c^2 \begin{pmatrix} A^+ A & 0 \\ 0 & A A^+ \end{pmatrix} \equiv \begin{pmatrix} H^\uparrow & 0 \\ 0 & H^\downarrow \end{pmatrix}$$

is a super-hamiltonian.

The energy degeneracy discussed in 6.6._DiracElectronsInMagneticField.pdf can therefore be understood in terms of SUSY.