7.2. Classical Fields

Classical field $\phi^{cl}(x)$ is the expectation of a quantum field $\phi(x)$ with respect to some state $| f \rangle$, i.e.,

$$\phi^{cl}(x) = \langle f | \phi(x) | f \rangle \qquad x = (t, \boldsymbol{r})$$

Let

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x} a_{\boldsymbol{k}}$$

An eigenstate $| f \rangle$ of $a_{\boldsymbol{k}}$ is called a coherent state (see 1.5._CoherentStatesAndVonNeumannLattice.pdf) :

$$a_{\boldsymbol{k}} | f \rangle = \alpha_{\boldsymbol{k}} | f \rangle$$

where

$$\alpha_{\boldsymbol{k}} = \sqrt{n_{\boldsymbol{k}}}\, e^{i\theta_{\boldsymbol{k}}} \qquad n_{\boldsymbol{k}} = \alpha^*_{\boldsymbol{k}} \alpha_{\boldsymbol{k}}$$

$$\therefore \qquad \phi^{cl}(x) = \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x} \langle f | a_{\boldsymbol{k}} | f \rangle$$

$$= \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x} \alpha_{\boldsymbol{k}}$$

$$= \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x + i\theta_k} \sqrt{n_{\boldsymbol{k}}}$$

$$\to \qquad \phi(x) | f \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x} a_{\boldsymbol{k}} \left| f \right\rangle$$

$$= \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot x + i\theta_k} \sqrt{n_{\boldsymbol{k}}} \left| f \right\rangle$$

$$= \phi^{cl}(x) \left| f \right\rangle$$

Let $| \underline{0} \rangle$ be the false vacuum, i.e.,

$$\langle \underline{0} | \phi(x) | \underline{0} \rangle = 0 \qquad \langle \underline{0} | \underline{0} \rangle = 1$$

while the true ground state is

$$\langle 0 | \phi(x) | 0 \rangle = v \neq 0 \qquad \langle 0 | 0 \rangle = 1$$

The classical field (or vacuum) associated with $| 0 \rangle$ is therefore

$$\phi^{cl}(x) = v$$

Theorem

The coherent state $| f \rangle$ associated with a given classical field

$$\phi^{cl}(x) = \langle f | \phi(x) | f \rangle$$

can be obtained from the false vacuum by a unitary tranform,

$$| f \rangle = e^{-i Q(t)} \left| \underline{0} \right\rangle$$

where

$$Q(t) = \frac{1}{\hbar} \int d^3 r\, \phi^{cl}(x)\, \pi(x) \;= Q^+(t)$$

& $\pi(t, \boldsymbol{r})$ is the conjugate momentum of $\phi(t, \boldsymbol{r})$, i.e.,

$$[\phi(t, \boldsymbol{r}), \pi(t, \boldsymbol{r}')] = i\hbar\, \delta(\boldsymbol{r} - \boldsymbol{r}')$$

Proof

$$Q(t) = \frac{1}{\hbar} \int d^3 r\, \phi^{\text{cl}}(x)\, \pi(x)$$

$$\rightarrow \quad [Q(t), \phi(t, \boldsymbol{r})] = \frac{1}{\hbar} \int d^3 r'\, \phi^{\text{cl}}(t, \boldsymbol{r}')\, [\pi(t, \boldsymbol{r}'), \phi(t, \boldsymbol{r})]$$

$$= -i\, \phi^{\text{cl}}(t, \boldsymbol{r})$$

$$\therefore \quad e^{i Q(t)}\, \phi(x)\, e^{-i Q(t)} = \phi(x) + [i Q(t), \phi(x)] + \frac{1}{2!}[i Q(t), [i Q(t), \phi(x)]] + \ldots$$

$$= \phi(x) + \phi^{\text{cl}}(x)$$

where $\left[Q(t), \phi^{\text{cl}}(x) \right] = 0$ since $\phi^{\text{cl}}(x)$ is not an operator.

By definition,

$$\langle \underline{0} \mid \phi(x) \mid \underline{0} \rangle = 0$$

$$\rightarrow \quad \left\langle \underline{0} \right| e^{i Q(t)}\, \phi(x)\, e^{-i Q(t)} \left| \underline{0} \right\rangle = \left\langle \underline{0} \right| \phi^{\text{cl}}(x) \left| \underline{0} \right\rangle = \phi^{\text{cl}}(x)$$

$$\phi^{\text{cl}}(x) = \langle f \mid \phi(x) \mid f \rangle \quad \rightarrow \quad \mid f \rangle = e^{-i Q(t)} \left| \underline{0} \right\rangle$$

QED

$\mid 0 \rangle$

For the classical vacuum

$$\phi^{\text{cl}}(x) = \langle 0 \mid \phi(x) \mid 0 \rangle = v$$

one sets

$$Q(t) = v \frac{1}{\hbar} \int d^3 r\, \pi(t, \boldsymbol{r})$$

$$\rightarrow \quad \mid 0 \rangle = e^{-i Q(t)} \left| \underline{0} \right\rangle$$

Goldstone Mode

For each topological sector, value of $\phi^{\text{cl}}(x) = v$ is obtained by minimizing the classical potential energy (H is also minimized by setting $K.E. = 0$). In some sectors, this local energy minimum may be higher than the actual ground state energy. But, as explained earlier, the corresponding states are still stable since there's an infinite energy barrier between sectors.

For small fluctuations around $\phi^{\text{cl}}(x)$, let

$$\phi(x) = \phi^{\text{cl}}(x) + \eta(x)$$

If there exists a continuous range of η that leaves the energy unchanged, there is a gapless (Goldstone) mode (see. 4.7._GoldstoneTheorem.pdf).

For example, consider a system with translational invariance.

If $\phi^{\text{cl}}(x)$ minimizes H, so does $\phi^{\text{cl}}(t, \boldsymbol{r} - \boldsymbol{r}_0)$ for arbitrary $\boldsymbol{r}_0$.

Thus, fluctuations

$$\phi(x) = \phi^{\text{cl}}(t, \boldsymbol{r} - \boldsymbol{r}_0) \simeq \phi^{\text{cl}}(t, \boldsymbol{r}) + (\boldsymbol{r} - \boldsymbol{r}_0)\, \phi^{\text{cl}\,\prime}(t, \boldsymbol{r}_0)$$

$$\equiv \phi^{\text{cl}}(x) + \eta(x)$$

will give rise to a Goldstone mode. The broken symmetry here is caused by the system settling in a state with a particular $\boldsymbol{r}_0$.