7.3. Solitary Waves, Kinks & Solitons

Note: All fields in this section are classical.

Solitary wave = Localized wave that travels with constant shape, size, & speed.
(The wave speed usually depends only on wave size & shape.)

Soliton = Solitary wave that is particle-like, i.e., maintains its shape, size, & hence speed, after a collision with other solitons.

Kink = topologically stabilized solitary wave (not a soliton).

Mathematically, solitonary waves are exact solutions to non-linear differential eqs.

Korteweg–de Vries (KdV) Eq.

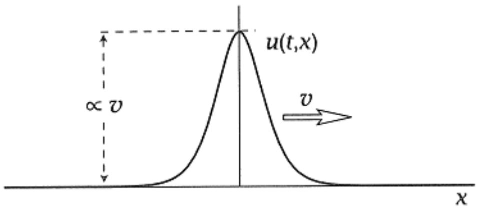
KdV eq. is

$$\partial_t u + \alpha\, u\, \partial_x u + \beta\, \partial_{xxx} u = 0 \qquad (\ \alpha,\ \beta \text{ real constants })$$

Solution is the KdV solitary wave

$$u(t,\ x) = \frac{3\, v}{\alpha} \operatorname{sech}^2\left[\frac{1}{2}\sqrt{\frac{v}{\beta}}\ (x - v\, t)\right] \qquad (\ v = \text{speed })$$

which is a non-topological soliton.



KdV soliton

Real Klein-Gordon (K-G) Eq.

Real K-G ϕ^4 Hamiltonian (see 4.2._RealKlein-GordonField.pdf):

$$\mathcal{H} = \frac{1}{2} f \left[\frac{1}{c^2} (\partial_t \phi)^2 + (\partial_x \phi)^2 + \frac{g}{2} (\phi^2 - v^2)^2\right]$$

→ 2 degenerate vacua with $\phi = \pm v$.

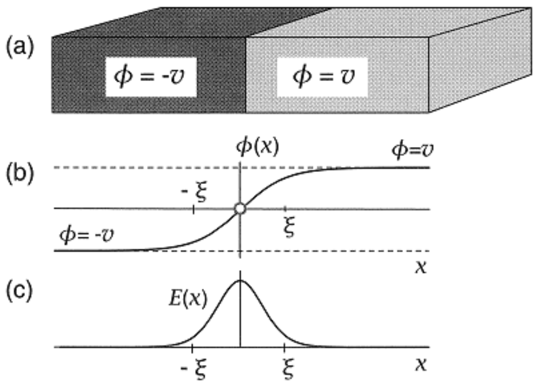
A kink at rest & centered at $x = 0$ is given by (see L._1-DSolitonSolutions.pdf)

$$\phi_{\text{kink}}(x) = v \tanh \frac{x}{2\,\xi} \qquad \xi = \frac{1}{v}\sqrt{\frac{1}{2\, g}}$$

$$\phi_{\text{kink}}(\pm\infty) = \pm v$$

with an energy density

$$E(x) = \frac{1}{2} f\, g\, v^4 \operatorname{sech}^4\left(\frac{x}{2\,\xi}\right)$$



(*a*) 2 vacua

(*b*) Kink

(*c*) Energy density

An anti-kink has $\phi = v$ on the left side & $\phi = -v$ on the right.
Hence, with the interface at $x = 0$,

$$\phi_{\text{antikink}}(x) = \phi_{\text{kink}}(-x)$$

Taking a Lorentz boost to a frame travelling with velocity u, we have

$$\phi_{\text{kink}}(t,\, x) = v \tanh\left(\frac{1}{2\,\xi} \frac{x - u\,t}{\sqrt{1 - (u/c)^2}}\right)$$

with $1 > u/c > -1$.

Kink is stabilized by topology
(it connects 2 vacua & can’t be deformed to a single vacuum without violating E-finiteness).

Kink is not a soliton (not particle-like) since 2 kinks can’t exist side by side (only kink & anti-kink can).