7.4. Sine-Gordon Solitons

Topological soliton that can be rigorously quantized.

Model for

Parallel magnetic flux penetrating Josephson junction.
Bilayer QH system.
Edge excitations in QH system.

Real K-G eq. is (see 3.3._RealKlein-GordonField.pdf)

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

(1+1)-D version:

$$\mathcal{L} = \frac{1}{2} f \left[\left(\frac{1}{c} \partial_t \phi \right)^2 - (\partial_x \phi)^2 - \frac{1}{\lambda^2} \phi^2 \right] \qquad \frac{1}{\lambda} = \frac{m c}{\hbar}$$

E-L eq.:

$$\frac{1}{c^2} \partial_{t t} \phi - \partial_{x x} \phi + \frac{1}{\lambda^2} \phi = 0$$

Sine-Gordon model :

$$\mathcal{L} = \frac{1}{2} f \left[\left(\frac{1}{c} \partial_t \phi \right)^2 - (\partial_x \phi)^2 - \frac{2}{\lambda^2} (1 - \cos\phi) \right]$$

(Ezawa uses $f = \hbar c$)

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = f \frac{1}{c^2} \partial_t \phi \qquad \frac{\partial \mathcal{L}}{\partial \partial_x \phi} = -f \partial_x \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -f \frac{1}{\lambda^2} \sin\phi$$

E-L eq.:

$$\frac{1}{c^2} \partial_{t t} \phi - \partial_{x x} \phi + + \frac{1}{\lambda^2} \sin\phi = 0 \quad \to \qquad \text{K-G eq. for } |\phi| << 1$$

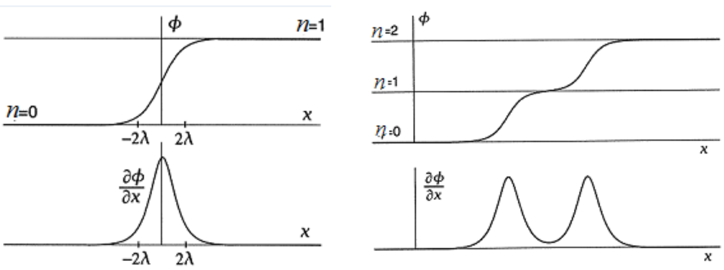
$$\mathcal{H} = \pi \partial_t \phi - \mathcal{L}$$

$$= \frac{1}{2} f \left[(\partial_t \phi)^2 + (\partial_x \phi)^2 + \frac{2}{\lambda^2} (1 - \cos\phi) \right]$$

Classical vacuum:

$$\frac{\partial \mathcal{H}}{\partial \phi} = 0 = f \frac{1}{\lambda^2} \sin\phi \qquad \to \qquad \phi = 2 \pi n \qquad n = 0, \pm 1, \ \ldots$$

$\mathcal{L}$ is invariant under $\phi \to \phi + 2 \pi n$.
Symmetry is broken spontaneously when a particular vacuum is chosen.



A kink centered at $x = x_0$ & rises from $2 n \pi$ to $2(n+1)\pi$ is given by

$$\phi(x) = 4\tan^{-1}\left[\exp\left(\frac{x - x_0}{\lambda}\right)\right] + 2\,n\,\pi \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

As explained in L._1-DSolitonSolutions.pdf, only such kinks, of height 2π, can be called solitons.

Its energy is

$$E_{SG} = \frac{1}{2} f \int_{-\infty}^{\infty} d\,x \left[(\partial_x \phi)^2 + \frac{2}{\lambda^2} (1 - \cos\phi) \right] = f\,\frac{8}{\lambda}$$

Lorentz boost to a moving frame gives a soliton moving with velocity v as

$$\phi_v\,(x,\ t) = 4\tan^{-1}\left[\exp\left(\frac{x - x_0 - v\,t}{\lambda_v}\right)\right] \qquad \lambda_v = \lambda\sqrt{1 - (v/c)^2}$$

(See L._1-DSolitonSolutions.pdf. for proof that this indeed satisfies the E-L eq.).

The energy of ϕ_v is

$$E_v = 4\,f\,\frac{2 - v^2/c^2}{\lambda_v}$$

$$= \frac{8\,f}{\lambda_v} - \frac{4\,f\,v^2/c^2}{\lambda_v}$$

A solution that describes 2 solitons colliding at $t = 0$ & then move apart is given by

$$\phi_2(x,\ t) = 4\tan^{-1}\left(\frac{v}{c}\,\frac{\sinh\left(\frac{x}{\lambda_v}\right)}{\cosh\left(\frac{v\,t}{\lambda_v}\right)} \right)$$

Proof that this is indeed a solution is rather tedious but can be done easily using the computer software *Mathematica*. (see SGKinks.nb, which also contains a simulation of the motion depicted in ϕ_2.)

In general,

Topological kink → Trivially conserved topological current independent of $\mathcal{H}$.

1-D: $$J^\mu(t,\ x) = \frac{1}{2\pi}\,\epsilon^{\mu\nu}\,\partial_\nu\,\phi \qquad \epsilon^{01} = -\epsilon^{10} = 1 \quad , \qquad \epsilon^{00} = \epsilon^{11}$$

→ $$\partial_\mu J^\mu = \frac{1}{2\pi}\,\epsilon^{\mu\nu}\,\partial_\mu \partial_\nu\,\phi = 0$$

Topological charge:

$$Q = \int_{-\infty}^{\infty} d\,x\,J^0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\,x\,\partial_x\phi = \frac{1}{2\pi}[\,\phi(\infty) - \phi(-\infty)]$$

Since the height of our solitons is 2π, we have

$Q = 0$ for any vacuum

$Q = \pm 1$ for kink / anti-kinks.

$Q = \pm 2$ for 2-kink / 2-anti-kinks.

...

Q labels topological sectors.