7.5.a. Flux Quantization

Complex KG field interacting with EM field in Higgs potential (see 5.4._Anderson-HiggsMechanism.pdf) :

$$\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} = f\left[\left(\partial^\mu - i\,q^*\,A^\mu\right)\phi^+ \cdot \left(\partial_\mu + i\,q^*\,A_\mu\right)\phi - \frac{g}{2}\left(\phi^+\,\phi - v^2\right)^2\right]$$

where $q^* = \dfrac{q}{\hbar\,c}$.

For a static magnetic field,

$$\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} = f\left[\left(\partial^k - i\,q^*\,A^k\right)\phi^+ \cdot \left(\partial_k + i\,q^*\,A_k\right)\phi - \frac{g}{2}\left(\phi^+\,\phi - v^2\right)^2\right]$$

$$= f\left[-\left(\nabla + i\,q^*\,\boldsymbol{A}\right)\phi^+ \cdot \left(\nabla - i\,q^*\,\boldsymbol{A}\right)\phi - \frac{g}{2}\left(\phi^+\,\phi - v^2\right)^2\right]$$

$$\mathcal{H}_{\text{EM}} = \frac{1}{8\,\pi\,\mu_m}\,\boldsymbol{B}^2$$

$$\rightarrow \qquad \mathcal{H}_{\text{matter}} + \mathcal{H}_{\text{int}} = f\left[\left(\nabla + i\,q^*\,\boldsymbol{A}\right)\phi^+ \cdot \left(\nabla - i\,q^*\,\boldsymbol{A}\right)\phi + \frac{g}{2}\left(\phi^+\,\phi - v^2\right)^2\right]$$

Note that by setting

$$f \rightarrow \frac{\hbar^2}{2\,m} \qquad \& \qquad f\,g \rightarrow g$$

$\mathcal{H}_{\text{matter}} + \mathcal{H}_{\text{int}}$ for the K-G field becomes the one for the Schrodinger field.
(see 5.6.a._SpontaneousBrokenSymmetryForCooperPairs.pdf)

Consider a Bose condensate (superconductor).
Then, in the absence of $\boldsymbol{B}$, $|\,\phi\,|^2 = v^2$ everywhere.

When we turn on an external field $\boldsymbol{B}$, the London eq says that $\boldsymbol{B}$ can penentrate only a small distance into the condensate.

We therefore seek a solution such that for $r \rightarrow \infty$,

$$\boldsymbol{B} = 0$$
$$\left(\nabla - i\,q^*\,\boldsymbol{A}\right)\phi = 0$$
$$|\,\phi\,|^2 = v^2$$

which gives, for $r \rightarrow \infty$,

$$\phi(\boldsymbol{r}) = v\,e^{i\,f(\boldsymbol{r})} \qquad (\,f = \text{real}\,)$$

$$\boldsymbol{A}\,(\boldsymbol{r}) = -i\,\frac{1}{q^*}\,\frac{\nabla\phi}{\phi} = \frac{1}{q^*}\,\nabla f$$

The region around $\boldsymbol{r} = 0$ in which $\boldsymbol{B} \neq 0$ is called a vortex.

For the vacuum sector, $f(\boldsymbol{r}) = \text{const.}$
Regular function $f(\boldsymbol{r})$ can be gauged away.
∴ Only multi-valued $f(\boldsymbol{r})$ are non-trivial.
(ϕ & ∇f must be single-valued since they're physical quantities.)

The easiest way to construct such a field is in Cylindrical coordinates $(\rho,\ \theta,\ z)$ so that

$$\lim_{r\rightarrow\infty} \phi(\boldsymbol{r}) \equiv \phi^\infty(\theta) = v\,e^{i\,f(\theta)}$$

where θ is the azimuthal angle &

$$f(2\,\pi) = f(0) + 2\,\pi\,n \qquad n = \text{integers}$$

The simplest choice is

$$f(\theta) = n\,\theta \qquad \text{which implies} \qquad f(0) = 0$$

Working backwards, a field that satisfies

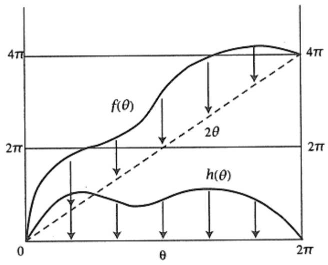
$$\phi^{\infty}(\theta) = v\, e^{i n \theta}$$

is

$$\phi(\boldsymbol{r}) \equiv \left| \phi(\boldsymbol{r}) \right| e^{i f(\boldsymbol{r})} \qquad (\, f = \text{real}\,)$$

$$= \left| \phi(\boldsymbol{r}) \right| e^{i n \theta + i h(\boldsymbol{r})} \qquad \text{where} \qquad h(\boldsymbol{r}) = f(\boldsymbol{r}) - n\,\theta$$

n labels topological sectors.



Relation between f, h & $n\,\theta$ for $n = 2$.

θ is not defined at $\boldsymbol{r} = 0$.
Hence, $\phi(\boldsymbol{r})$ is well-defined at $\boldsymbol{r} = 0$ only if $\phi(\boldsymbol{0}) = 0$ [i.e., a false vacuum (non-condensate) at the vortex center].
$U(1)$ symmetry broken by the Higgs potential is thus restored at the center.

Using

$$\theta = \tan^{-1} \frac{y}{x} \qquad \frac{d \tan^{-1} x}{d\, x} = \frac{1}{1 + x^2}$$

we have

$$\partial_x \theta = \frac{-y / x^2}{1 + (y/x)^2} = -\frac{y}{\rho^2} \qquad \rho^2 = x^2 + y^2$$

$$\partial_y \theta = \frac{1/x}{1 + (y/x)^2} = \frac{x}{\rho^2}$$

i.e., $$\partial_i \theta = -\frac{1}{\rho^2}\, \varepsilon_{i j 3}\, x_j$$

$\therefore$ $$\nabla \theta = \frac{1}{\rho^2} (-y,\ x,\ 0) = \frac{1}{\rho} \hat{\boldsymbol{\theta}}$$

Hence, for $\rho \to \infty$,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{q^*} \nabla f = \frac{n}{q^*} \nabla \theta = \frac{n}{q^* \rho} \hat{\boldsymbol{\theta}}$$

or $$\boldsymbol{A}_i = -\frac{n}{q^*} \frac{1}{\rho^2}\, \varepsilon_{i j 3}\, x_j$$

The flux through a surface S is

$$\Phi = \int d\, \boldsymbol{S} \cdot \boldsymbol{B} = \oint_C d\, \boldsymbol{r} \cdot \boldsymbol{A}$$

where C goes counterclockwise along the boundary of S.

Let S be an open surface that spans the entire $\boldsymbol{B} \neq 0$ region, then C lies entirely in the $\rho \to \infty$ region.

$$\Phi = \frac{n}{q^*} \oint_C d\, \boldsymbol{r} \cdot \nabla \theta = \frac{n}{q^*} \lim_{\rho \to \infty} \int_0^{2\pi} \rho\, d\,\theta\, \hat{\theta} \cdot \hat{\theta} \frac{1}{\rho} \frac{\partial \theta}{\partial \theta}$$

$$= \frac{n}{q^*} \int_0^{2\pi} d\,\theta = \frac{2 \pi n}{q^*}$$

$= n\,\Phi_0$

i.e., total flux through a vortex is quantized in units of

$$\Phi_0 = \frac{2\pi}{q^*} = \frac{2\pi\hbar c}{q} = \begin{cases} \text{Dirac flux unit for } q = e \\ \text{London flux unit for } q = 2e \end{cases}$$

Note: As a unit of measurement, $\Phi_0 > 0$ so that $|\,q\,|$ should be used in its definition. However, ± signs will then have to be added to our formulae to handle ± charges.

The (conserved) topological current is

$$J^\mu_{\text{vor}} = \frac{1}{\Phi_0}\,\varepsilon^{\mu\nu\lambda 3}\,\partial_\nu A_\lambda$$

$$\rightarrow \qquad J^0_{\text{vor}} = \frac{1}{\Phi_0}\,\varepsilon^{0\,\nu\lambda 3}\,\partial_\nu A_\lambda \quad = \frac{1}{\Phi_0}\,\varepsilon^{i\,j\,3}\,\partial_i A_j = \frac{1}{\Phi_0}\,\boldsymbol{B}_3$$

The topological charge is therefore

$$Q_{\text{vor}} = \int d^2 r\, J^0_{\text{vor}} = \frac{1}{\Phi_0}\int_S d\,\boldsymbol{S}\cdot\boldsymbol{B} \qquad \text{where } S \text{ is parallel to xy-plane}$$

$$= n$$

Thus, the topological sectors are classified by the number of flux unit the vortex carries.