## 7.5.b. Small Fluctuations

### Singular Gauge Transformation

For small fluctuations about the classical vacuum, let

$$\phi(\mathbf{r}) = [v + \eta(\mathbf{r})]\, e^{i n \theta}$$

$$\mathbf{A}(\mathbf{r}) = \mathbf{U}(\mathbf{r}) + \frac{n}{q^*} \nabla\theta = \mathbf{U}(\mathbf{r}) + \frac{n\,\Phi_0}{2\pi} \nabla\theta$$

which is not a genuine gauge transformation since, as will be proved below,

$$F_{ij}(\mathbf{r}) = \partial_i A_j - \partial_j A_i = -\partial_i \mathbf{A}_j + \partial_j \mathbf{A}_i$$

$$= -\partial_i \mathbf{U}_j + \partial_j \mathbf{U}_i + \frac{n}{q^*}\left(-\partial_i \partial_j \theta + \partial_j \partial_i \theta\right)$$

is singular at $\mathbf{r} = 0$.

Wherever $\theta$ is well defined,

$$-\partial_i \partial_j \theta + \partial_j \partial_i \theta = 0$$

The point $\mathbf{r} = 0$ requires special treatment.

Using

$$\partial_x \theta = -\frac{y}{\rho^2} \qquad\qquad \partial_y \theta = \frac{x}{\rho^2}$$

we have

$$\partial_y \partial_x \theta = -\frac{1}{\rho^2} + \frac{y^2}{\rho^4} \qquad\qquad \partial_x \partial_y \theta = \frac{1}{\rho^2} - \frac{x^2}{\rho^4}$$

$$\to \qquad \left(\partial_x \partial_y - \partial_y \partial_x\right)\theta = \frac{2}{\rho^2} - \frac{x^2 + y^2}{\rho^4} = \frac{1}{\rho^2}$$

Since

$$\int d^2 r\, \frac{1}{\rho^2} = \int_0^{2\pi} d\theta = 2\pi$$

$$\therefore \qquad \left(\partial_x \partial_y - \partial_y \partial_x\right)\theta = \frac{1}{2\pi}\, \delta^2(\mathbf{r})$$

Hence

$$F_{ij}(\mathbf{r}) = -\partial_i \mathbf{U}_j + \partial_j \mathbf{U}_i - \frac{n}{2\pi q^*}\, \varepsilon_{ij3}\, \delta^2(\mathbf{r})$$

is singular at $\mathbf{r} = 0$ as forewarned.

### $\mathbf{U}$, $\eta$ Hamiltonian

We now substitute

$$\phi(\mathbf{r}) = [v + \eta(\mathbf{r})]\, e^{i n \theta}$$

$$\mathbf{A}(\mathbf{r}) = \mathbf{U}(\mathbf{r}) + \frac{n}{q^*} \nabla\theta$$

into

$$\mathcal{H} = f\left[ (\nabla + i q^* \mathbf{A})\, \phi^+ \cdot (\nabla - i q^* \mathbf{A})\, \phi + \frac{g}{2}\left(\phi^+ \phi - v^2\right)^2 \right] + \frac{1}{8\pi \mu_m} \mathbf{B}^2$$

$$\nabla \phi(\mathbf{r}) = [\, i n (v + \eta) \nabla\theta + \nabla\eta\,]\, e^{i n \theta}$$

$$(\nabla - i q^* \mathbf{A})\, \phi = \left[ i n (v + \eta) \nabla\theta + \nabla\eta - i q^* \left( \mathbf{U} + \frac{n}{q^*} \nabla\theta \right)(v + \eta) \right] e^{i n \theta}$$

$$= [\nabla\eta - i\, q^* \boldsymbol{U}(v+\eta)]\, e^{i n\theta}$$

$\rightarrow$ $\quad (\nabla + i\, q^* \boldsymbol{A})\, \phi^+ = [\nabla\eta + i\, q^* \boldsymbol{U}(v+\eta)]\, e^{-i n\theta}$

$\therefore$ $\quad (\nabla + i\, q^* \boldsymbol{A})\, \phi^+ \cdot (\nabla - i\, q^* \boldsymbol{A})\, \phi = [\nabla\eta + i\, q^* \boldsymbol{U}(v+\eta)] \cdot [\nabla\eta - i\, q^* \boldsymbol{U}(v+\eta)]$

$$= \nabla\eta \cdot \nabla\eta + q^{*2}\, \boldsymbol{U}^2 (v+\eta)^2$$

$$= -\eta\, \nabla^2 \eta + v^2 q^{*2}\, \boldsymbol{U}^2 + O$$

where

$$O = \nabla \cdot (\eta \nabla \eta) + q^{*2}\, \boldsymbol{U}^2\, \eta\, (2\, v + \eta)$$

The divergence term can be dropped since it doesn't contribute to $H$.
The rest of $O$ is dropped in the lowest order of approximation.

$$\left(\phi^+ \phi - v^2\right)^2 = \left(2\, v\, \eta + \eta^2\right)^2 \simeq 4\, v^2\, \eta^2$$

$$\boldsymbol{B}_i = \varepsilon_{i\,j\,k}\, \partial_j\, \boldsymbol{A}_k$$

$$\boldsymbol{B}^2 = \varepsilon_{i\,j\,k}\, \partial_j\, \boldsymbol{A}_k\, \varepsilon_{i\,l\,m}\, \partial_l\, \boldsymbol{A}_m$$

$$= \left(\delta_{j\,l}\, \delta_{k\,m} - \delta_{j\,m}\, \delta_{k\,l}\right) \partial_j\, \boldsymbol{A}_k\, \partial_l\, \boldsymbol{A}_m$$

$$= \partial_j\, \boldsymbol{A}_k\, \partial_j\, \boldsymbol{A}_k - \partial_j\, \boldsymbol{A}_k\, \partial_k\, \boldsymbol{A}_j$$

$$= \partial_j \left(\boldsymbol{A}_k\, \partial_j\, \boldsymbol{A}_k\right) - \boldsymbol{A}_k\, \partial_j \partial_j\, \boldsymbol{A}_k - \partial_j \left(\boldsymbol{A}_k\, \partial_k\, \boldsymbol{A}_j\right) + \boldsymbol{A}_k\, \partial_k\, \partial_j\, \boldsymbol{A}_j$$

$$= -\boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} + \boldsymbol{A} \cdot \nabla (\nabla \cdot \boldsymbol{A}) + \partial_j \left(\boldsymbol{A} \cdot \partial_j\, \boldsymbol{A} - \boldsymbol{A} \cdot \nabla\, \boldsymbol{A}_j\right)$$

Dropping the divergence term & working in the Coulomb gauge ( $\nabla \cdot \boldsymbol{A} = 0$ ), we have

$$\boldsymbol{B}^2 = -\boldsymbol{A} \cdot \nabla^2 \boldsymbol{A}$$

$$= -\boldsymbol{U} \cdot \nabla^2 \boldsymbol{U}$$

$\therefore$ To 2nd order of $\boldsymbol{U}$ & $\eta$,

$$\mathcal{H} \simeq f\left(-\eta\, \nabla^2 \eta + v^2\, q^{*2}\, \boldsymbol{U}^2 + 2\, g\, v^2\, \eta^2\right) - \frac{1}{8\,\pi\,\mu_m}\, \boldsymbol{U} \cdot \nabla^2 \boldsymbol{U}$$

## Eqs. for $\boldsymbol{U}$ & $\eta$

To minimize $\mathcal{H}$,

$$\left.\frac{\delta \mathcal{H}}{\delta \phi}\right|_{\boldsymbol{A}=0} = \left.\frac{\delta \mathcal{H}}{\delta \eta}\right|_{\boldsymbol{U}=0} = 0 \quad \rightarrow \quad -2\, \nabla^2 \eta + 4\, g\, v^2\, \eta = 0$$

$$\left.\frac{\delta \mathcal{H}}{\delta \boldsymbol{A}}\right|_{\phi=0} = \left.\frac{\delta \mathcal{H}}{\delta \boldsymbol{U}}\right|_{\eta=0} = 0 \quad \rightarrow \quad -\frac{1}{4\,\pi\,\mu_m}\, \nabla^2 \boldsymbol{U} + 2\, f\, v^2\, q^{*2}\, \boldsymbol{U} = 0$$

Reminder: $\quad \dfrac{\delta \mathcal{H}}{\delta \phi} = \dfrac{\partial \mathcal{H}}{\partial \phi} - \nabla \cdot \dfrac{\partial \mathcal{H}}{\partial \nabla \phi} + \partial_i \partial_j \dfrac{\partial \mathcal{H}}{\partial\, \partial_i \partial_j \phi} - \ldots$

$\therefore$ $\quad \xi^2\, \nabla^2 \eta - \eta = 0 \quad$ with $\quad \xi^2 = \dfrac{1}{2\, g\, v^2}$

$$\lambda^2\, \nabla^2 \boldsymbol{U} - \boldsymbol{U} = 0 \text{ with } \quad \lambda^2 = \frac{1}{8\,\pi\, f\, q^{*2}\, v^2\, \mu_m} = \frac{\hbar^2 c^2}{8\,\pi\, f\, q^2\, v^2\, \mu_m}$$

where $\xi$ & $\lambda$ are the coherence & penetration lenths, respectively.
Note that they're the same as those given in 5.4._Anderson-HiggsMechanism.pdf .

## Solutions

$$\lambda^2\, \nabla^2 \boldsymbol{U} - \boldsymbol{U} = 0$$

is just the London eq., which was solved in 5.6.e._Vortices.pdf.

Adapting the results there, we have

$$\left( \frac{d^2}{d z^2} + \frac{1}{z} \frac{d}{d z} \right) B_z = \frac{1}{\lambda^2} B_z$$

$$\rightarrow \qquad B_z(\rho) = n \frac{1}{q^* \lambda^2} K_0\!\left(\frac{\rho}{\lambda}\right) = n \frac{\Phi_0}{2\pi\lambda^2} K_0\!\left(\frac{\rho}{\lambda}\right)$$

$$\boldsymbol{U}_i = -\lambda^2 \varepsilon_{ijk} \partial_j \boldsymbol{B}_k$$

$$= n \frac{1}{q^* \lambda} K_1\!\left(\frac{\rho}{\lambda}\right) \varepsilon_{ij3} \frac{x^j}{\rho} = n \frac{\Phi_0}{2\pi\lambda} K_1\!\left(\frac{\rho}{\lambda}\right) \varepsilon_{ij3} \frac{x^j}{\rho}$$

$$\boldsymbol{A}_i = \boldsymbol{U}_i + \frac{n}{q^*} \partial_i \theta = \left[ n \frac{\rho}{q^* \lambda} K_1\!\left(\frac{\rho}{\lambda}\right) - \frac{n}{q^*} \right] \varepsilon_{ij3} \frac{x^j}{\rho^2}$$

$$= \frac{n}{q^*} \left[ \frac{\rho}{\lambda} K_1\!\left(\frac{\rho}{\lambda}\right) - 1 \right] \varepsilon_{ij3} \frac{x^j}{\rho^2}$$

$$= n \frac{\Phi_0}{2\pi} \left[ \frac{\rho}{\lambda} K_1\!\left(\frac{\rho}{\lambda}\right) - 1 \right] \varepsilon_{ij3} \frac{x^j}{\rho^2}$$

$$\hat{\boldsymbol{\theta}} = \frac{1}{\rho}(-y, x, 0) \qquad \varepsilon_{ij3} x^j = (y, -x, 0)$$

$$\rightarrow \qquad \boldsymbol{A} = n \frac{\Phi_0}{2\pi\rho} \left[ 1 - \frac{\rho}{\lambda} K_1\!\left(\frac{\rho}{\lambda}\right) \right] \hat{\boldsymbol{\theta}}$$

∴ For $C_\rho$ = circle of radius $\rho$ centered at the origin,

$$\Phi(\rho) = \oint_{C_\rho} d\boldsymbol{r} \cdot \boldsymbol{A} = n \Phi_0 \left[ 1 - \frac{\rho}{\lambda} K_1\!\left(\frac{\rho}{\lambda}\right) \right]$$

Assuming all fields are independent of $z$, we have, in cylindrical coord.,

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}$$

Symmetry of the system suggests

$$\eta(\boldsymbol{r}) = \eta(\rho)$$

so that

$$\phi(\boldsymbol{r}) = [v + \eta(\rho)] e^{i n \theta}$$

&

$$\frac{d^2 \eta}{d \rho^2} + \frac{1}{\rho} \frac{d\eta}{d\rho} - \frac{\eta}{\xi^2} = 0$$

which is just the eq satisfied by $B_z$ so that a solution that goes to 0 at infinity is

$$K_0\!\left(\frac{\rho}{\xi}\right) \underset{\rho \to \infty}{\rightarrow} \sqrt{\frac{\xi}{\rho}}\, e^{-\rho/\xi}$$

However, $K_0$ is singular at $\rho = 0$. Thus, a solution that is regular at both $\rho = 0$ & $\infty$, while being differentiable everywhere, requires using $K_0$ for $\rho > \rho_0$ & $I_0$ for $\rho < \rho_0$ such that at $\rho_0$, both their values & slopes match.

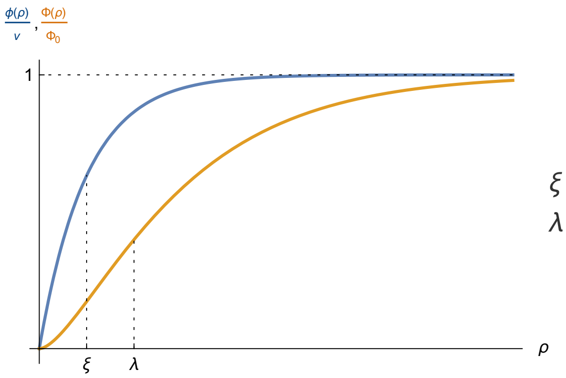
If we're only interested in an approximate solution for $\phi$ that satisfies exactly the boundary conditions

$$\phi(\boldsymbol{r}) = \begin{cases} 0 & \rho = 0 \\ 1 & \rho = \infty \end{cases}$$

then

$$\phi(\boldsymbol{r}) \simeq v\left(1 - e^{-\rho/\xi}\right) e^{i n \theta}$$

Qualitative behavior of the normalized field $\frac{\phi(\rho)}{v}$ & flux $\frac{\Phi(\rho)}{\Phi_0}$:




$\xi$ = coherence length
$\lambda$ = penetration length