7.6.a. Homotopy Group

Ref: R.Aldrovandi, J.G.Pereira, "An Introduction to Geometrical Physics", Chap.3.

The following is a working description of the homotopy groups using minimal mathematics. See ref for details.

Two continuous functions (or mappings)

$$f, g : X \to Y$$

that map between two topological spaces X & Y are homotopic if they can be continuously deformed into each other.

Given X & Y, all functions that are homotopic to each other are said to belong to the same homotopy class. It follows that every function must belong to one & only one homotopy class.

The n^{th} homotopy group $\pi_n(X)$ uses as elements the classes of the functions

$$f : S^n \to X$$

Consider the 1st homotopy group (also called the fundamental group) $\pi_1(X)$. Its classes are obtained by considering how many non-equivalent (non-homotopic) ways to draw (map continuously) a closed loop (S^1) on X.

Group multiplication is just the combination of loops that share a common contact point.

If X is simply connected (no holes or other "defects" in it), all loops can be continuously shrinked to a point (all f are homotopic to the identity map). Thus, the group contains only the identity element, i.e., $\pi_n(X) = 1$ or $\pi_n(X) = 0$ (the identity of an abelian group is 0 since its group composition is called addition).

For example, $\pi_1(\mathbb{R}^n) = 0$.

Let X has a single hole in it that can prevent a loop to shrink continuously to a point. Then all loops that encircle the hole n times belong to the n^{th} class, where $n = 0, \pm 1, \pm 2,$ ...., with the $\pm$ sign denoting the sense of traversing the loop. Hence, $\pi_1(X) = \mathbb{Z}$, the group of the integers.

Another example is $\pi_1(S^1) = \mathbb{Z}$

It's easy to see that a loop on the surface of a sphere can always contract continuously to a point. Hence, $\pi_1(S^2) = 0$.

Similarly, one may suggest

$$\pi_n(S^m) = 0 \qquad \forall \; n < m$$

However, rigorous treatment of $\pi_n(X)$ with $n \geq 2$ requires the concept of covering space, into which we shall not venture (see ref.).