7.6.b. Application to Vortex

At the boundary ($\rho \to \infty$), the field ϕ of a vortex becomes (see 7.5.a._FluxQuantization.pdf)

$$\lim_{\rho \to \infty} \phi(\boldsymbol{r}) \equiv \phi^{\infty}(\theta) = v\, e^{i n \theta}$$

Thus, ϕ^{∞} maps a unit circle in the xy plane to a circle of radius v in the complex plane, i.e.,

$$\phi^{\infty} : \quad S^1 \;\to\; S^1$$

Since the associated fundamental group is

$$\pi_1\left(S^1\right) = \mathbb{Z}$$

$\phi(\boldsymbol{r})$ can be labeled by the integer n.

Put it another way, $\phi(\boldsymbol{r})$ is classified by its topolgical charge n, which is a topological property that indicates how many times it goes around the origin as the $\boldsymbol{r}$ goes around it once.

Transition between ϕ of different n is forbidden since they belong to different topological sectors. Thus, only vortices carrying quantized flux are stable.

Alternatively, one can interpret $\phi(\boldsymbol{r})$ as the super-state wave function so that the z-axis, where $\phi = 0$, is in the normal-state.

The associated group is then $\pi_1(X) = \mathbb{Z}$, where X is $\mathbb{R}^3$ with the z-axis taken out.