7.6.c. Skyrmions

Consider the $O(N)$ nonlinear sigma model (see 4.4._SigmaModel.pdf)

$$\mathcal{L} = \frac{1}{2} f \ \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} \qquad \left(\text{Ezawa: } f = J_s \ \& \ \eta_{\mu\nu} = -g_{\mu\nu}^{\text{Ezawa}} \right)$$

$$= \frac{1}{2} f \left(\frac{1}{c^2} \dot{\boldsymbol{\phi}}^2 - \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} \right)$$

where

$$\boldsymbol{\phi} = (\phi_1, \ \ldots, \ \phi_N)$$

with

$$\sum_{a=1}^{N} \phi_a^2 = \boldsymbol{\phi} \cdot \boldsymbol{\phi} = \boldsymbol{\phi}^2 = 1$$

$$\therefore \qquad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\phi}}} \cdot \dot{\boldsymbol{\phi}} - \mathcal{L}$$

$$= \frac{1}{2} f \left(\frac{1}{c^2} \dot{\boldsymbol{\phi}}^2 + \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} \right)$$

For static fields

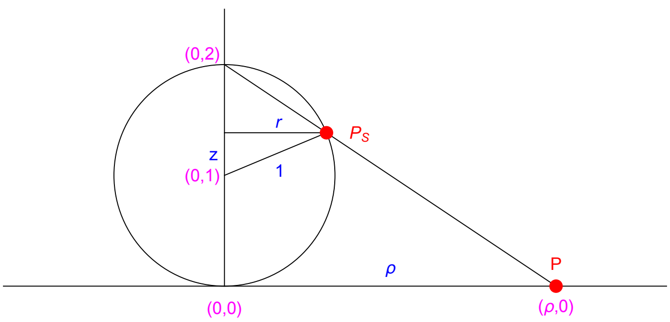
$$H = \frac{1}{2} f \int d^m r \ \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}$$

The integral is finite only if the integrand goes to 0 as $r \to \infty$.
This is guaranteed if

$$\lim_{r \to \infty} \boldsymbol{\phi}(\mathbf{r}) = \boldsymbol{\phi}^\infty$$

where $\boldsymbol{\phi}^\infty$ is a constant vector. Thus, to each ϕ_a, all points at infinity are the same, i.e., the configuration space $\mathbb{R}^m$ is compactified into a spherical surface S^m.

For example, the xy plane $\mathbb{R}^2$ is compactified into the unit spherical surface S^2 so that all points at infinity are mapped to a single point, say, the north pole, on S^2. (This is analogous to the use of the Riemann sphere to compactify the complex plane.)



With reference to the graph above, a point $P = (\rho, 0)$ on the xy-plane is mapped to a point $P_S = (r, 1 + z)$ on the sphere. Using

$$\frac{\rho}{2} = \frac{r}{1 - z} \qquad \& \qquad z^2 + r^2 = 1$$

we get

$$r = \sqrt{1 - z^2} \qquad \to \qquad \frac{\rho}{2} = \sqrt{\frac{1 + z}{1 - z}}$$

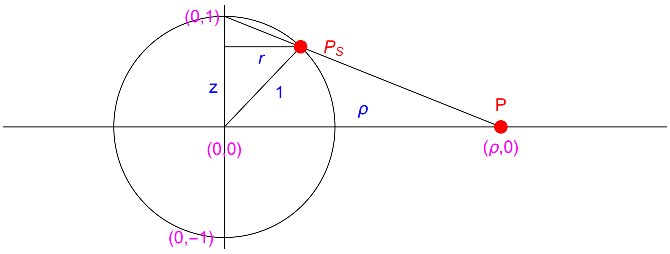
$$\therefore \qquad z = \frac{\rho^2 - 4}{\rho^2 + 4}$$

$$\rightarrow \qquad 1 + z = \frac{2\rho^2}{\rho^2 + 4} \qquad \& \qquad r = \frac{4\rho}{\rho^2 + 4}$$

i.e., the mapping is

$$(\rho, 0) \quad \rightarrow \quad \left(\sqrt{1 - z^2}, 1 + z\right) = \frac{2\rho}{\rho^2 + 4} (2, \rho)$$

Alternatively, we can place the sphere center at $(0, 0)$ so that points with $\rho > 1$ is mapped to the upper hemisphere ($z > 0$), & points with $\rho < 1$ to the lower one ($z < 0$).



Setting $P_S = (r, z)$, we have

$$\rho = \frac{r}{1 - z} \qquad \& \qquad z^2 + r^2 = 1$$

so that

$$r = \sqrt{1 - z^2} \qquad \rightarrow \qquad \rho = \sqrt{\frac{1 + z}{1 - z}}$$

$$\therefore \qquad z = \frac{\rho^2 - 1}{\rho^2 + 1} \qquad \& \qquad r = \frac{2\rho}{\rho^2 + 1}$$

i.e., the mapping is

$$(\rho, 0) \quad \rightarrow \quad \left(\sqrt{1 - z^2}, z\right) = \frac{1}{\rho^2 + 1} \left(2\rho, \rho^2 - 1\right)$$

To get Ezawa's results (with allowance for error), set $\rho = r_{\text{Ezawa}}$.

Now, $\boldsymbol{\phi}^\infty$ is a vector on S^{N-1}. The boundary of an m-D system is S^m.
The boundary condition $\lim_{r \to \infty} \boldsymbol{\phi}(\boldsymbol{r}) = \boldsymbol{\phi}^\infty$ thus maps S^m to S^{N-1}.

The homotopy group is therefore $\pi_m\left(S^{N-1}\right)$.

If it's nontrivial, i.e., $\pi_m\left(S^{N-1}\right) \neq 0$, there'll be topological solitons called skyrmions.

Note that the argument also apply to the anisotropic nonlinear sigma model

$$\mathcal{L} = \frac{1}{2} f \sum_a c_a \partial_\mu \phi_a \cdot \partial^\mu \phi_a$$