7.6.d. Scaling Theorem

As discussed near the end of 7.6.c._Skyrmions.pdf, topological solitons exist in the $O(N)$ nonlinear sigma model in m-D space if $\pi_m(S^{N-1})$ is nontrivial, i.e., $N = m + 1$.

The energies of these solitons are given for static fields by

$$H = \frac{1}{2} f \int d^m r \; \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}$$

Let $\boldsymbol{\phi}_0(\boldsymbol{r})$ gives a local energy minimum H_0.

Consider $\boldsymbol{\phi}(\boldsymbol{r}) = \boldsymbol{\phi}_0(\lambda \boldsymbol{r})$ with energy

$$H(\lambda) = \frac{1}{2} f \int d^m r \; \partial_i \boldsymbol{\phi}_0(\lambda \boldsymbol{r}) \cdot \partial_i \boldsymbol{\phi}_0(\lambda \boldsymbol{r})$$

Setting $\boldsymbol{r}' = \lambda \boldsymbol{r}$, we have

$$d^m r = \lambda^{-m} d^m r' \qquad \partial_i = \frac{\partial}{\partial \boldsymbol{r}_i} = \lambda \frac{\partial}{\partial \boldsymbol{r}_i'} = \lambda \partial_i'$$

$$\rightarrow \qquad H(\lambda) = \frac{1}{2\lambda^{m-2}} f \int d^m r \; \partial_i \boldsymbol{\phi}_0(\boldsymbol{r}) \cdot \partial_i \boldsymbol{\phi}_0(\boldsymbol{r})$$

$$= \lambda^{2-m} H(1) = \lambda^{2-m} H_0$$

Thus, except for $m = 2$, $H(\lambda)$ can be made arbitrarily small by changing λ, i.e.,

$$\lim_{\lambda \to 0} H(\lambda) = 0 \qquad \text{for } m = 1$$

$$\lim_{\lambda \to \infty} H(\lambda) = 0 \qquad \text{for } m \geq 3$$

Since $\partial_i \boldsymbol{\phi}_0(\boldsymbol{r}) \cdot \partial_i \boldsymbol{\phi}_0(\boldsymbol{r}) \geq 0$,

$$H = 0 \qquad \rightarrow \qquad \boldsymbol{\phi}_0(\boldsymbol{r}) = \text{const} \quad \text{(not a soliton)}$$

On the other hand, solitons are determined solely by boundary conditions so that they are invariant under the scaling operation discussed above. Hence, $O(N)$ solitons can exist only in 2-D space.

Together with the nontriviality requirement on $\pi_m(S^{N-1})$, we see that only $N \leq 3$ can support (2-D) topological solitons.

In order for $O(4)$ in 3-D to support topological solitons, H must be modified to break the scaling property. The one used by Skyrme to study π & σ mesons is

$$H = \frac{1}{2} f \int d^m r \; \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} + \kappa \int d^m r \left[(\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi})^2 - \sum_{i,j} (\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi})^2 \right]$$

where the κ term scales as λ^{4-m}.

The resultant solitons are called skyrmions