7.7.a. Topological Charge

Consider

$$\mathcal{L} = \frac{1}{2} f \ \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi}$$

where

$$\boldsymbol{\phi} = (\phi_1,\ \phi_2,\ \phi_3)$$
$$\boldsymbol{\phi}^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$$
$$\lim_{r \to \infty} \boldsymbol{\phi}(\boldsymbol{r}) = \boldsymbol{\phi}^\infty = (0,\ 0,\ 1)$$

In a planar geometry, consider the following 4-vector (or current)

$$J_s^\mu = \alpha\, \varepsilon^{\mu\nu\lambda 3}\, \varepsilon_{abc}\, \phi_a\, \partial_\nu \phi_b\, \partial_\lambda \phi_c$$

where α is a constant.

Since $\varepsilon^{\mu\nu\lambda 3}$ is totally anti-symmetric,

$$\varepsilon^{\mu\nu\lambda 3}\, \partial_\mu \partial_\nu \phi_b = \varepsilon^{\mu\nu\lambda 3}\, \partial_\mu \partial_\lambda \phi_c = 0$$

→ $$\partial_\mu J_s^\mu = \alpha\, \varepsilon^{\mu\nu\lambda 3}\, \varepsilon_{abc}\, \partial_\mu \phi_a\, \partial_\nu \phi_b\, \partial_\lambda \phi_c$$

$$\boldsymbol{\phi}^2 = 1 \qquad \to \qquad \sum_a \phi_a\, \partial_\mu \phi_a = 0$$

If $\phi_a \neq 0$, we have

$$\partial_\mu \phi_a = -\frac{1}{\phi_a} \sum_{d \neq a} \phi_d\, \partial_\mu \phi_d$$

If $\phi_a = 0$, we have

$$\partial_\mu \phi_b = -\frac{1}{\phi_b}\, \partial_\mu \phi_c \qquad \text{where } a \neq b \neq c$$

Consider now the summation over a in $\partial_\mu J_s^\mu$.

If $\phi_a \neq 0$,

$$d \neq a \quad \& \quad a \neq b \neq c, \quad \to \quad d = b \text{ or } c.$$

Hence, we have either

$$\varepsilon^{\mu\nu\lambda 3}\, \partial_\mu \phi_b\, \partial_\nu \phi_b = 0 \qquad \text{or} \qquad \varepsilon^{\mu\nu\lambda 3}\, \partial_\mu \phi_c\, \partial_\lambda \phi_c = 0$$

If $\phi_a = 0$, we have

$$\varepsilon^{\mu\nu\lambda 3}\, \partial_\nu \phi_c\, \partial_\lambda \phi_c = 0$$

→ $$\partial_\mu J_s^\mu = 0$$

i.e., j_s^μ is conserved.

Since it arises only from the geometrical property $\boldsymbol{\phi}^2 = 1$, j_s^μ is a topological current.

The topological charge is, by definition,

$$Q_s = \int d^2 r\, J_s^0 \ = \alpha \int d^2 r\ \varepsilon^{0ij3}\, \varepsilon_{abc}\, \phi_a\, \partial_i \phi_b\, \partial_j \phi_c$$

when α is chosen to make Q_s a dimensionless integer, Q_s is called the Pontryagin number.