7.7.b. General Solutions for Skyrmions

Equations for $\boldsymbol{\phi}$

$$Q_s = \alpha \int d^2 r\; \varepsilon^{0ij3}\, \varepsilon_{abc}\, \phi_a\, \partial_i \phi_b\, \partial_j \phi_c$$

Consider the quantity

$$\psi_{ia}^{(\pm)} = \partial_i \phi_a \pm \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_j \phi_c$$

where everything is real.

For a fixed a, it is the i^{th} component of a spatial vector so that

$$\psi_{ia}^{(\pm)}\, \psi_{i\underline{a}}^{(\pm)} \ge 0$$

where the underlined $\underline{a}$ means that a is exempted from any implicit summation.

Summing over a, we have

$$\psi_{ia}^{(\pm)}\, \psi_{ia}^{(\pm)} \ge 0$$

i.e., $\left(\partial_i \phi_a \pm \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_j \phi_c \right) \left(\partial_i \phi_a \pm \varepsilon_{0ik3}\, \varepsilon_{ade}\, \phi_d\, \partial_k \phi_e \right) \ge 0$

where we've used $\varepsilon^{0ij3} = \varepsilon_{0ij3}$.

Considering the 4 terms in the product separately, we have

$$\partial_i \phi_a\, \partial_i \phi_a = \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}$$

$$\varepsilon_{0ik3}\, \varepsilon_{ade}\, \phi_d\, \partial_i \phi_a\, \partial_k \phi_e = \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_i \phi_a\, \partial_j \phi_c$$

$$\begin{aligned}
&\varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_j \phi_c\, \varepsilon_{0ik3}\, \varepsilon_{ade}\, \phi_d\, \partial_k \phi_e \\
&\qquad = \delta_{jk} \left(\delta_{bd}\, \delta_{ce} - \delta_{be}\, \delta_{cd} \right) \phi_b\, \phi_d\, \partial_j \phi_c\, \partial_k \phi_e && (j, k = 1, 2) \\
&\qquad = \delta_{jk} \left(\phi_b\, \phi_b\, \partial_j \phi_c\, \partial_k \phi_c - \phi_b\, \phi_c\, \partial_j \phi_c\, \partial_k \phi_b \right) \\
&\qquad = \delta_{jk}\, \partial_j \phi_c\, \partial_k \phi_c && (\, \phi_a\, \partial_j \phi_a = 0 \,) \\
&\qquad = \partial_j \phi_c\, \partial_j \phi_c = \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}
\end{aligned}$$

$\rightarrow$ $\partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} \ge \mp \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_i \phi_a\, \partial_j \phi_c$

$$\begin{aligned}
\int d^2 r\; \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} &\ge \mp \int d^2 r\; \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_i \phi_a\, \partial_j \phi_c \\
&= \pm \int d^2 r\; \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_a\, \partial_i \phi_b\, \partial_j \phi_c
\end{aligned}$$

$$H = \frac{1}{2} f \int d^2 r\; \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}$$

$\rightarrow$ $H \ge \pm \dfrac{f}{2\alpha} Q_s$

Thus, for each topological sector specified by Q_s, the energy minimum is given by

$$H = \pm \frac{f}{2\alpha} Q_s$$

i.e., $\psi_{ia}^{(\pm)}\, \psi_{ia}^{(\pm)} = 0$

$$\psi_{ia}^{(\pm)} = 0$$

$$\partial_i \phi_a \pm \varepsilon_{0ij3}\, \varepsilon_{abc}\, \phi_b\, \partial_j \phi_c = 0$$

which consists of $2 \times 2 \times 3 = 12$ eqs.

For $a = 1$ and 2, we have $2 \times 2 \times 2 = 8$ eqs

$$\partial_i \phi_1 \pm \varepsilon_{0ij3} \left(\phi_2\, \partial_j \phi_3 - \phi_3\, \partial_j \phi_2 \right) = 0$$

$$\partial_i \phi_2 \pm \varepsilon_{0ij3} \left(\phi_3 \partial_j \phi_1 - \phi_1 \partial_j \phi_3 \right) = 0$$

ω

Setting

$$\Phi = \phi_1 - i \phi_2 \qquad \rightarrow \qquad i \Phi = i \phi_1 + \phi_2$$

the 8 real eqs for $a = 1$ and 2 can be combined into $2 \times 2 = 4$ complex eqs.

$$\partial_i \Phi \pm i \varepsilon_{0ij3} \left(\Phi \partial_j \phi_3 - \phi_3 \partial_j \Phi \right) = 0$$

ε_{0ik3} times the whole eq gives

$$\varepsilon_{0ik3} \partial_i \Phi \pm i \delta_{jk} \left(\Phi \partial_j \phi_3 - \phi_3 \partial_j \Phi \right) = 0$$

$$\rightarrow \qquad \Phi \partial_k \phi_3 - \phi_3 \partial_k \Phi = \pm i \varepsilon_{0ik3} \partial_i \Phi$$

$$\Phi \partial_k \phi_3 = \phi_3 \partial_k \Phi \mp i \varepsilon_{0ki3} \partial_i \Phi$$

Consider the field

$$\omega = \frac{\Phi}{1 - \phi_3}$$

$$\rightarrow \qquad \partial_i \omega = \frac{(1 - \phi_3) \partial_i \Phi + \Phi \partial_i \phi_3}{(1 - \phi_3)^2}$$

$$= \frac{(1 - \phi_3) \partial_i \Phi + \phi_3 \partial_i \Phi \mp i \varepsilon_{0ij3} \partial_j \Phi}{(1 - \phi_3)^2}$$

$$= \frac{\partial_i \Phi \mp i \varepsilon_{0ij3} \partial_j \Phi}{(1 - \phi_3)^2}$$

Thus,

$$\partial_x \omega = \frac{\partial_x \Phi \mp i \partial_y \Phi}{(1 - \phi_3)^2}$$

$$\partial_y \omega = \frac{\partial_y \Phi \pm i \partial_x \Phi}{(1 - \phi_3)^2} = \pm i \partial_x \omega$$

Setting

$$z = x + i y \qquad\qquad z^* = x - i y$$

$$\rightarrow \qquad x = \frac{1}{2} (z + z^*) \qquad\qquad y = \frac{1}{2i} (z - z^*)$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial z^*} = \frac{\partial x}{\partial z^*} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z^*} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Since $H \geq 0$,

$$H = \pm \frac{f}{2 \alpha} Q_s = \frac{f}{2 \alpha} \left| Q_s \right|$$

Taking the upper sign, which corresponds to $Q_s > 0$ & skyrmions, we have

$$\frac{\partial \omega}{\partial z^*} = \frac{1}{2} \left(\frac{\partial_x \Phi - i \partial_y \Phi}{(1 - \phi_3)^2} + i \frac{\partial_y \Phi + i \partial_x \Phi}{(1 - \phi_3)^2} \right) = 0$$

$$\rightarrow \qquad \omega = \omega(z)$$

i.e., any analytic function is a solution.

Taking the lower sign, which corresponds to $Q_s < 0$ & anti-skyrmions, we have

$$\frac{\partial \omega}{\partial z} = \frac{1}{2} \left(\frac{\partial_x \Phi + i \partial_y \Phi}{(1 - \phi_3)^2} - i \frac{\partial_y \Phi - i \partial_x \Phi}{(1 - \phi_3)^2} \right) = 0$$

→ $\omega = \omega(z^*)$

i.e., any function obtained by replacing z by z^* in an analytic function is a solution.

Solutions

Once ω is chosen, the skyrmion field can be obtained as follows.

$$\omega = \frac{\Phi}{1-\phi_3} \qquad \Phi = \phi_1 - i\,\phi_2$$

$$\rightarrow \quad |\,\omega\,|^2 = \frac{\phi_1^2+\phi_2^2}{(1-\phi_3)^2} = \frac{1-\phi_3^2}{(1-\phi_3)^2} = \frac{1+\phi_3}{1-\phi_3}$$

$$\therefore \quad \phi_3 = \frac{|\,\omega\,|^2 - 1}{|\,\omega\,|^2 + 1} \qquad 1-\phi_3 = \frac{2}{|\,\omega\,|^2+1}$$

$$\Phi = \omega\,(1-\phi_3) = \frac{2\,\omega}{|\,\omega\,|^2+1}$$

$$\phi_1 = \frac{1}{2}(\Phi+\Phi^*) = \frac{\omega + \omega^*}{|\,\omega\,|^2+1}$$

$$\phi_2 = \frac{1}{2}\,i\,(\Phi-\Phi^*) = i\,\frac{\omega - \omega^*}{|\,\omega\,|^2+1}$$

By definition, an analytic function has a Taylor series expansion. Writing

$$z = \rho\, e^{i\,\theta} \qquad \rightarrow \qquad z^n = \rho^n\, e^{i\,n\,\theta} \qquad z^{*\,n} = \rho^n\, e^{-i\,n\,\theta}$$

Thus, to adapt a skyrmion solution to an anti-skyrmion one, set

$$\rho^n \rightarrow \rho^n \qquad \& \qquad n\,\theta \rightarrow -n\,\theta$$