7.7.d. Solutions in Polar Form

$\boldsymbol{\phi}$ can be written in polar form using 2 real fields ϑ & σ such that

$$\phi_1(\boldsymbol{r}) = \sqrt{1-\sigma^2(\boldsymbol{r})}\ \cos\vartheta(\boldsymbol{r})$$
$$\phi_2(\boldsymbol{r}) = -\sqrt{1-\sigma^2(\boldsymbol{r})}\ \sin\vartheta(\boldsymbol{r})$$
$$\phi_3(\boldsymbol{r}) = \sigma(\boldsymbol{r})$$

For the examples discussed in 7.7.c._Examples.pdf, $\vartheta(\boldsymbol{r}) = \pm n\,\theta(\boldsymbol{r})$.
Thus, as θ goes from 0 to 2π on the xy-plane, $\boldsymbol{\phi}$ sweeps the S^2 manifold $\pm n$ times.

We now re-write

$$J_s^\mu = \alpha\,\varepsilon^{\mu\nu\lambda 3}\,\varepsilon_{abc}\,\phi_a\,\partial_\nu\phi_b\,\partial_\lambda\phi_c$$

& $$Q_s = \alpha\int d^2r\ \varepsilon^{0ij3}\,\varepsilon_{abc}\,\phi_a\,\partial_i\phi_b\,\partial_j\phi_c$$

in terms of ϑ & σ.

To begin,

$$\partial_\mu\phi_1 = -\sqrt{1-\sigma^2}\ \sin\vartheta\,\partial_\mu\vartheta - \frac{\sigma}{\sqrt{1-\sigma^2}}\cos\vartheta\,\partial_\mu\sigma$$
$$\partial_\mu\phi_2 = -\sqrt{1-\sigma^2}\ \cos\vartheta\,\partial_\mu\vartheta + \frac{\sigma}{\sqrt{1-\sigma^2}}\sin\vartheta\,\partial_\mu\sigma$$
$$\partial_\mu\phi_3 = \partial_\mu\sigma$$

→ $$\varepsilon_{1bc}\,\phi_1\,\partial_\nu\phi_b\,\partial_\lambda\phi_c = \phi_1(\partial_\nu\phi_2\,\partial_\lambda\phi_3 - \partial_\nu\phi_3\,\partial_\lambda\phi_2)$$
$$= \phi_1\left[\left(-\sqrt{1-\sigma^2}\ \cos\vartheta\,\partial_\nu\vartheta + \frac{\sigma}{\sqrt{1-\sigma^2}}\sin\vartheta\,\partial_\nu\sigma\right)\partial_\lambda\sigma \right.$$
$$\left. - \left(-\sqrt{1-\sigma^2}\ \cos\vartheta\,\partial_\lambda\vartheta + \frac{\sigma}{\sqrt{1-\sigma^2}}\sin\vartheta\,\partial_\lambda\sigma\right)\partial_\nu\sigma\right]$$
$$= \phi_1\sqrt{1-\sigma^2}\ \cos\vartheta\ (\partial_\lambda\vartheta\,\partial_\nu\sigma - \partial_\nu\vartheta\,\partial_\lambda\sigma)$$
$$= (1-\sigma^2)\cos^2\vartheta\ (\partial_\lambda\vartheta\,\partial_\nu\sigma - \partial_\nu\vartheta\,\partial_\lambda\sigma)$$

$$\varepsilon_{2bc}\,\phi_2\,\partial_\nu\phi_b\,\partial_\lambda\phi_c = \phi_2(\partial_\nu\phi_3\,\partial_\lambda\phi_1 - \partial_\nu\phi_1\,\partial_\lambda\phi_3)$$
$$= \phi_2\left[\left(-\sqrt{1-\sigma^2}\ \sin\vartheta\,\partial_\lambda\vartheta - \frac{\sigma}{\sqrt{1-\sigma^2}}\cos\vartheta\,\partial_\lambda\sigma\right)\partial_\nu\sigma \right.$$
$$\left. - \left(-\sqrt{1-\sigma^2}\ \sin\vartheta\,\partial_\nu\vartheta - \frac{\sigma}{\sqrt{1-\sigma^2}}\cos\vartheta\,\partial_\nu\sigma\right)\partial_\lambda\sigma\right]$$
$$= \phi_2\sqrt{1-\sigma^2}\ \sin\vartheta\,(-\partial_\lambda\vartheta\,\partial_\nu\sigma + \partial_\nu\vartheta\,\partial_\lambda\sigma)$$
$$= (1-\sigma^2)\sin^2\vartheta\,(\partial_\lambda\vartheta\,\partial_\nu\sigma - \partial_\nu\vartheta\,\partial_\lambda\sigma)$$

$$\varepsilon_{3bc}\,\phi_3\,\partial_\nu\phi_b\,\partial_\lambda\phi_c = \phi_3(\partial_\nu\phi_1\,\partial_\lambda\phi_2 - \partial_\nu\phi_2\,\partial_\lambda\phi_1)$$
$$= \phi_3\left[\left(-\sqrt{1-\sigma^2}\ \sin\vartheta\,\partial_\nu\vartheta - \frac{\sigma}{\sqrt{1-\sigma^2}}\cos\vartheta\,\partial_\nu\sigma\right)\right.$$
$$\times\left(-\sqrt{1-\sigma^2}\ \cos\vartheta\,\partial_\lambda\vartheta + \frac{\sigma}{\sqrt{1-\sigma^2}}\sin\vartheta\,\partial_\lambda\sigma\right)$$

$$= \phi_3 \left(-\sigma \sin^2 \vartheta\, \partial_\nu \vartheta\, \partial_\lambda \sigma + \sigma \cos^2 \vartheta\, \partial_\lambda \vartheta\, \partial_\nu \sigma \right.$$
$$\left. - \sigma \cos^2 \vartheta\, \partial_\nu \vartheta\, \partial_\lambda \sigma + \sigma \sin^2 \vartheta\, \partial_\lambda \vartheta\, \partial_\nu \sigma \right)$$
$$= \phi_3\ \sigma\, (\partial_\lambda \vartheta\, \partial_\nu \sigma - \partial_\nu \vartheta\, \partial_\lambda \sigma)$$
$$= \sigma^2\, (\partial_\lambda \vartheta\, \partial_\nu \sigma - \partial_\nu \vartheta\, \partial_\lambda \sigma)$$

→ $$J_s^\mu = \alpha\, \varepsilon^{\mu\nu\lambda 3} (\partial_\lambda \vartheta\, \partial_\nu \sigma - \partial_\nu \vartheta\, \partial_\lambda \sigma)$$

$$Q_s = \alpha \int d^2 r\ \varepsilon^{0 i j 3} \left(\partial_j \vartheta\, \partial_i \sigma - \partial_i \vartheta\, \partial_j \sigma\right)$$
$$= 2\,\alpha \int d^2 r\, (\partial_2 \vartheta\, \partial_1 \sigma - \partial_1 \vartheta\, \partial_2 \sigma)$$
$$= 2\,\alpha \int d\,x \wedge d\,y\ \frac{\partial\,(\sigma, \vartheta)}{\partial\,(x, y)}$$

where

$d\,x \wedge d\,y$ = oriented surface element of the xy-plane

$$\frac{\partial\,(\sigma, \vartheta)}{\partial\,(x, y)} = \begin{vmatrix} \partial_1 \sigma & \partial_1 \vartheta \\ \partial_2 \sigma & \partial_2 \vartheta \end{vmatrix}$$ = Jacobian of the transformation $(x, y) \to (\sigma, \vartheta)$

∴ $$Q_s = 2\,\alpha \int d\,\sigma \wedge d\,\vartheta$$
$$= 2\,\alpha\ n\left(\text{surface area of the unit sphere } S^2\right)$$
$$= 8\,\pi\,\alpha\,n$$

Setting $Q_s = n$

→ $$\alpha = \frac{1}{8\,\pi}$$