7.7.e. Examples Revisited

The example discussed in 7.7.c._Examples.pdf is already in polar form:

$$\phi_1 = \frac{2\rho^n \kappa^n}{\rho^{2n}+\kappa^{2n}} \cos n\theta \qquad \phi_2 = -\frac{2\rho^n \kappa^n}{\rho^{2n}+\kappa^{2n}} \sin n\theta \qquad \phi_3 = \frac{\rho^{2n}-\kappa^{2n}}{\rho^{2n}+\kappa^{2n}}$$

Comparing with the polar form in 7.7.d._SolutionsInPolarForm.pdf,

$$\phi_1 = \sqrt{1-\sigma^2}\,\cos\vartheta \qquad \phi_2 = -\sqrt{1-\sigma^2}\,\sin\vartheta \qquad \phi_3 = \sigma$$

we have

$$\sigma = \frac{\rho^{2n}-\kappa^{2n}}{\rho^{2n}+\kappa^{2n}}$$

$$\vartheta = \pm n\,\theta \qquad \text{for} \quad \begin{matrix}\text{skyrmions}\\ \text{antiskyrmions}\end{matrix} \qquad \text{with} \quad n = 1, 2, \ldots$$

Using (see 7.5.a._FluxQuantization.pdf)

$$\partial_x \rho = \frac{x}{\rho} = \cos\theta \qquad \partial_y \rho = \frac{y}{\rho} = \sin\theta$$

$$\partial_x \theta = -\frac{y}{\rho^2} = -\frac{1}{\rho}\sin\theta \qquad \partial_y \theta = \frac{x}{\rho^2} = \frac{1}{\rho}\cos\theta$$

$$\rightarrow \qquad \partial_x = \cos\theta\,\partial_\rho - \frac{1}{\rho}\sin\theta\,\partial_\theta \qquad \partial_y = \sin\theta\,\partial_\rho + \frac{1}{\rho}\cos\theta\,\partial_\theta$$

we have

$$\partial_x \vartheta = \mp n \frac{1}{\rho}\sin\theta \qquad \partial_y \vartheta = \pm n \frac{1}{\rho}\cos\theta$$

Q_s

$$J_s^0 = \frac{1}{8\pi}\varepsilon^{0ij3}\left(\partial_j\vartheta\,\partial_i\sigma - \partial_i\vartheta\,\partial_j\sigma\right)$$

$$= \frac{1}{4\pi}\left(\partial_2\vartheta\,\partial_1\sigma - \partial_1\vartheta\,\partial_2\sigma\right)$$

$$= \frac{1}{4\pi}\left[\pm n\frac{1}{\rho}\cos\theta\left(\cos\theta\,\partial_\rho\sigma - \frac{1}{\rho}\sin\theta\,\partial_\theta\sigma\right) \pm n\frac{1}{\rho}\sin\theta\left(\sin\theta\,\partial_\rho\sigma + \frac{1}{\rho}\cos\theta\,\partial_\theta\sigma\right)\right]$$

$$= \pm\frac{n}{4\pi\rho}\partial_\rho\sigma$$

$$\rightarrow \qquad Q_s = \int d^2r\, J_s^0 = \pm\frac{n}{4\pi}\int_0^{2\pi} d\theta \int_0^\infty \rho\, d\rho\, \frac{1}{\rho}\partial_\rho\sigma$$

$$= \pm\frac{n}{4\pi}\int_0^{2\pi} d\theta\,\left[\sigma(\infty) - \sigma(0)\right]$$

Caution: In general, $\sigma(\mathbf{r}) = \sigma(\rho, \theta)$ so the θ integral can't be evaluated beforehand.

From 7.7.c._Examples.pdf ,

$$\sigma(\infty) = 1 \qquad \sigma(0) = -1$$

$\rightarrow \qquad Q_s = \pm n \qquad$ as advertised.

H

From 7.7.b._GeneralSolutionsForSkyrmions.pdf ,

$$H = \frac{1}{2} f \int d^2 r \; \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}$$

while the minimal energy in sector n is

$$H = \pm 4\pi f \, Q_s = 4\pi f \; | \, n \, |$$

We now try to evaluate the H integral directly using the polar form solutions.

$$\phi_1 = \sqrt{1-\sigma^2} \cos\vartheta \qquad \phi_2 = -\sqrt{1-\sigma^2} \sin\vartheta \qquad \phi_3 = \sigma$$

$$\to \quad \partial_i \phi_1 = \phi_1 \left(\frac{\sigma}{1-\sigma^2} \partial_i \sigma - \tan\vartheta \, \partial_i \vartheta \right)$$

$$\partial_i \phi_1 \, \partial_i \phi_1 = \phi_1^2 \left(\frac{\sigma}{1-\sigma^2} \nabla\sigma - \tan\vartheta \nabla\vartheta \right) \cdot \left(\frac{\sigma}{1-\sigma^2} \nabla\sigma - \tan\vartheta \nabla\vartheta \right)$$

$$= \phi_1^2 \left(\frac{\sigma^2}{(1-\sigma^2)^2} \nabla\sigma \cdot \nabla\sigma + \tan^2\vartheta \nabla\vartheta \cdot \nabla\vartheta - \frac{2\sigma \tan\vartheta}{1-\sigma^2} \nabla\sigma \cdot \nabla\vartheta \right)$$

$$= \frac{\sigma^2 \cos^2\vartheta}{1-\sigma^2} \nabla\sigma \cdot \nabla\sigma + (1-\sigma^2) \sin^2\vartheta \nabla\vartheta \cdot \nabla\vartheta - 2\sigma \sin\vartheta \cos\vartheta \nabla\sigma \cdot \nabla\vartheta \Big)$$

$$\partial_i \phi_2 = \phi_2 \left(\frac{\sigma}{1-\sigma^2} \partial_i \sigma + \cot\vartheta \, \partial_i \vartheta \right)$$

$$\partial_i \phi_2 \, \partial_i \phi_2 = \phi_2^2 \left(\frac{\sigma}{1-\sigma^2} \nabla\sigma + \cot\vartheta \nabla\vartheta \right) \cdot \left(\frac{\sigma}{1-\sigma^2} \nabla\sigma + \cot\vartheta \nabla\vartheta \right)$$

$$= \phi_2^2 \left(\frac{\sigma^2}{(1-\sigma^2)^2} \nabla\sigma \cdot \nabla\sigma + \cot^2\vartheta \nabla\vartheta \cdot \nabla\vartheta + \frac{2\sigma \cot\vartheta}{1-\sigma^2} \nabla\sigma \cdot \nabla\vartheta \right)$$

$$= \frac{\sigma^2 \sin^2\vartheta}{1-\sigma^2} \nabla\sigma \cdot \nabla\sigma + (1-\sigma^2) \cos^2\vartheta \nabla\vartheta \cdot \nabla\vartheta + 2\sigma \sin\vartheta \cos\vartheta \nabla\sigma \cdot \nabla\vartheta \Big)$$

$$\partial_i \phi_3 = \partial_i \sigma$$

$$\partial_i \phi_3 \, \partial_i \phi_3 = \nabla\sigma \cdot \nabla\sigma$$

$$\to \quad \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} = \partial_i \phi_a \, \partial_i \phi_a = \frac{1}{1-\sigma^2} \nabla\sigma \cdot \nabla\sigma + (1-\sigma^2) \nabla\vartheta \cdot \nabla\vartheta$$

$$= \frac{1}{1-\sigma^2} \nabla\sigma \cdot \nabla\sigma + n^2 (1-\sigma^2) \nabla\theta \cdot \nabla\theta$$

(For both skyrmions & anti-skyrmions.)

$$\therefore \quad H = \frac{1}{2} f \int d^2 r \left(\frac{1}{1-\sigma^2} \nabla\sigma \cdot \nabla\sigma + n^2 (1-\sigma^2) \nabla\theta \cdot \nabla\theta \right)$$

$$\sigma = \frac{\rho^{2n} - \kappa^{2n}}{\rho^{2n} + \kappa^{2n}}$$

$$\to \quad 1 - \sigma^2 = \frac{4 \kappa^{2n} \rho^{2n}}{(\rho^{2n} + \kappa^{2n})^2}$$

$$\nabla\sigma = \left(\frac{1}{\rho^{2n} + \kappa^{2n}} - \frac{\rho^{2n} - \kappa^{2n}}{(\rho^{2n} + \kappa^{2n})^2} \right) 2n \rho^{2n-1} \nabla\rho$$

$$= 4n \kappa^{2n} \frac{\rho^{2n-1}}{(\rho^{2n} + \kappa^{2n})^2} \hat{\rho}$$

$$= n\left(1 - \sigma^2\right) \frac{1}{\rho} \hat{\rho}$$

$$\therefore \qquad \frac{1}{1 - \sigma^2} \nabla\sigma \cdot \nabla\sigma = n \frac{1}{\rho} \hat{\rho} \cdot \nabla\sigma \; = n \frac{1}{\rho} \frac{\partial \sigma}{\partial \rho}$$

Also, taking $\hat{\rho}\cdot$ on the $\nabla\sigma$ equation gives

$$\frac{\partial \sigma}{\partial \rho} = n\left(1 - \sigma^2\right) \frac{1}{\rho}$$

$$\therefore \qquad \left(1 - \sigma^2\right) \nabla\theta \cdot \nabla\theta = \left(1 - \sigma^2\right) \frac{1}{\rho^2} \; = \frac{1}{n\,\rho} \frac{\partial \sigma}{\partial \rho}$$

Hence,

$$H = n f \int d^2 r \, \frac{1}{\rho} \frac{\partial \sigma}{\partial \rho}$$

$$= 4\,\pi\, n f$$ (see section Q_s)