7.8.a. Preliminaries

Ref: Frankel = [T.Frankel, "The Geometry of Physics", 2nd ed. (2004)]

We state without proof two theorems:

If G is simply connected,

$$\pi_2(G/H) = \pi_1(H) \qquad \text{(= denotes isomorphism)}$$

$$\pi_n(G \otimes G') = \pi_n(G) \oplus \pi_n(G')$$

[see V.L.Golo & A.M.Perelomov, Phys.Lett.B79,112(1978)]

(see Frankel §17.2 for the definition of the coset space G/H)

Note: We use the symbol G to stand for both the (topological) group G & the topological space (or manifold) formed by the elements of G. The exact meaning should be clear by context.

Let (see V.L.Golo)

$$\mathrm{CP}^{N-1} = \mathrm{SU}(N)/[\,U(1) \otimes \mathrm{SU}(N-1)\,] \qquad (\, N \ge 2 \,)$$

(see , Frankel, p.22, for definition of the complex projective space CP^N.)

Using (see 7.6.a._HomotopyGroup.pdf),

$$\pi_1[U(1)] = \pi_1(S^1) = \mathbb{Z}$$

& (see , Frankel §22.4),

$$\pi_1[\mathrm{SU}(N)] = 0 \qquad \forall\ N$$

we have

$$\begin{aligned}\pi_2(\mathrm{CP}^{N-1}) &= \pi_1[\,U(1) \otimes \mathrm{SU}(N-1)\,] \\ &= \pi_1[\,U(1)] \oplus \pi_1[\mathrm{SU}(N-1)\,] \\ &= \pi_1[\,U(1)] \ = \mathbb{Z}\end{aligned}$$

Note: The group manifold of $U(1)$ is S^1.

Thus, CP^{N-1} supports skyrmions for arbitrary $N \ge 2$.

It is called the CP^{N-1} sigma model.

CP^1 is locally isomorphic to $O(3)$.

CP^3 skyrmions describe charged excitations in bilayer QH systems.

A further generalization is called the Grassmannian $G_{N,k}$ model where

$$G_{N,k} = \mathrm{SU}(N)/[\,U(1) \otimes \mathrm{SU}(N-k)]$$

→ $\qquad \pi_2(G_{N,k}) = \mathbb{Z}$

The Grassmannian $G_{N,k}$ model will be used to describe the bilayer QH system with filling factor $\nu = 2$.

[see Frankel §17.2b for the definition of a (real) Grassmannian manifold $\mathrm{Gr}(k,\,N)$]