7.8.b. CP^{N-1} Sigma Model

Convention: ∂ operates only on its adjacent neighbor, i.e.,

$$\partial\, a\, b \equiv (\partial\, a)\, b \qquad\qquad a\, b\, \overleftarrow{\partial} \equiv a\, (\partial\, b)$$

$\mathcal{L}$

Consider N complex scalar fields represented by the normalized vector

$$\boldsymbol{n} = (n_1, \ \ldots, n_N)^T$$

with

$$\boldsymbol{n}^+(x)\, \boldsymbol{n}(x) = n_a^*(x)\, n_a(x) = 1$$

Let (c.f. 7.5.a._FluxQuantization.pdf)

$$\mathcal{L} = 2f \left(D^\mu \boldsymbol{n} \right)^+ D_\mu \boldsymbol{n}$$

where

$$D_\mu = \partial_\mu + i K_\mu \qquad\qquad D^\mu = \partial^\mu + i K^\mu$$

$$\begin{aligned}
\therefore \qquad \mathcal{L} &= 2f\, D^{\mu *}\, \boldsymbol{n}^+ D_\mu \boldsymbol{n} \\
&= 2f \left(\partial^\mu - i K^\mu \right) \boldsymbol{n}^+ \left(\partial_\mu + i K_\mu \right) \boldsymbol{n} \\
&= 2f \left(\partial^\mu - i K^\mu \right) n_a^* \left(\partial_\mu + i K_\mu \right) n_a \\
&= 2f \left(\partial_\mu - i K_\mu \right) n_a^* \left(\partial^\mu + i K^\mu \right) n_a \\
&= 2f\, D_\mu^*\, \boldsymbol{n}^+ D^\mu \boldsymbol{n} \\
&= 2f \left(D_\mu \boldsymbol{n} \right)^+ D^\mu \boldsymbol{n}
\end{aligned}$$

Note:

$$D^{\mu *} D_\mu = D_\mu^* D^\mu = \eta_{\mu\nu} D^{\mu *} D^\nu$$

but $\quad D^{\mu *}\, \boldsymbol{n}^+ D_\mu \boldsymbol{n} \neq D^\mu\, \boldsymbol{n}^+ D_\mu^* \boldsymbol{n}$

SU(N) Symmetry

Let U be an $N \times N$ unitary matrix of constant components.
Consider the transformation

$$\boldsymbol{n} \ \rightarrow\ \boldsymbol{n}' = U \boldsymbol{n} \qquad \& \qquad D_\mu \boldsymbol{n}' = U D_\mu \boldsymbol{n}$$

$$\Rightarrow \qquad \boldsymbol{n}^+ \ \rightarrow\ \boldsymbol{n}'^+ = \boldsymbol{n}^+ U^+ \qquad \& \qquad D_\mu \boldsymbol{n}'^+ = D_\mu \boldsymbol{n}^+ U^+$$

$$\begin{aligned}
\therefore \qquad \mathcal{L} \ \rightarrow\ \mathcal{L}' &= 2f \left(D^\mu \boldsymbol{n}' \right)^+ D_\mu \boldsymbol{n}' \\
&= 2f \left(D^\mu \boldsymbol{n} \right)^+ U^+ U D_\mu \boldsymbol{n} \\
&= 2f \left(D^\mu \boldsymbol{n} \right)^+ D_\mu \boldsymbol{n} \qquad\qquad (U^+ U = 1) \\
&= \mathcal{L}
\end{aligned}$$

i.e., $\mathcal{L}$ is invariant under the global SU(N) group.

The presence of the gauge field K_μ means that $\mathcal{L}$ is invariant under a local gauge transformation, thus promoting the symmetry to a local one.

E-L Eqs.

$$\frac{\partial \mathcal{L}}{\partial n_a} = 2f\, i K_\mu \left(\partial^\mu - i K^\mu \right) n_a^* = 2f\, i K_\mu D^{\mu *} n_a^*$$

or

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{n}} = 2f\, i K_\mu D^{\mu *} \boldsymbol{n}^+$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu n_a} = 2 f \left(\partial^\mu - i K^\mu \right) n_a^* = 2 f D^{\mu *} n_a^*$$

or $$\frac{\partial \mathcal{L}}{\partial \partial_\mu \boldsymbol{n}} = 2 f D^{\mu *} \boldsymbol{n}^+$$

∴ $$i K_\mu D^{\mu *} \boldsymbol{n}^+ - \partial_\mu D^{\mu *} \boldsymbol{n}^+ = 0$$

$$D_\mu^* D^{\mu *} \boldsymbol{n}^+ = 0$$

Either taking $^+$ of the above eq., or doing the variation w.r.t. $\boldsymbol{n}^+$, gives

$$D_\mu D^\mu \boldsymbol{n} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_\mu} = f\, i \left[-n_a^* \left(\partial^\mu + i K^\mu \right) n_a + \left(\partial^\mu - i K^\mu \right) n_a^* n_a \right]$$

$$= -f\, i \left(n_a^* \partial^\mu n_a - \partial^\mu n_a^* n_a + 2 i K^\mu n_a^* n_a \right)$$

$$= -f\, i \left(n_a^* \overleftrightarrow{\partial}^\mu n_a + 2 i K^\mu \right)$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\nu K_\mu} = 0$$

→ $$K^\mu = \frac{1}{2} i\, n_a^* \overleftrightarrow{\partial}^\mu n_a = \frac{1}{2} i\, \boldsymbol{n}^+ \overleftrightarrow{\partial}^\mu \boldsymbol{n}$$

Using

$$n_a^* \partial^\mu n_a - \partial^\mu n_a^* n_a = 2 n_a^* \partial^\mu n_a - \partial^\mu \left(n_a^* n_a \right)$$

$$= 2 n_a^* \partial^\mu n_a = 2 \boldsymbol{n}^+ \partial^\mu \boldsymbol{n}$$

→ $$K^\mu = i\, \boldsymbol{n}^+ \partial^\mu \boldsymbol{n}$$

Thus, K^μ is not an independent (or dynamic) field.

Number of Independent Fields

To begin, there're $2N$ real fields in $\boldsymbol{n}$.

$\boldsymbol{n}^+ \boldsymbol{n} = 1$ → Only $2N - 1$ at most are independent.

$\mathcal{L}$ has $U(1)$ local gauge symmetry, i.e., invariant under the gauge transformation

$$\boldsymbol{n}(x) \to \boldsymbol{n}(x)\, e^{i f(x)} \qquad K_\mu(x) \to K_\mu(x) + \partial_\mu f(x)$$

Given any 2 component fields n_a & n_b, we can always find a real $f(x)$ such that

$$n_b(x) = n_a(x)\, e^{i f(x)}$$

Hence, n_b can be replaced by n_a & f. In other word, of the 4 real fields in n_a & n_b only the 3 independent real fields in n_a & f are independent.

Note that this trick can be used only once since the gauge (i.e., f) can be fixed only once.

This means only $N - 1$ complex fields at most are independent in CP^{N-1}.

$\mathcal{H}$

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{n}_a} = 2 \frac{f}{c} D^{0 *} n_a^* \qquad \text{or} \qquad \boldsymbol{p} = 2 \frac{f}{c} D^{0 *} \boldsymbol{n}^+$$

$$p_a^* = \frac{\partial \mathcal{L}}{\partial \dot{n}_a^*} = 2 \frac{f}{c} D_0 n_a \qquad \text{or} \qquad \boldsymbol{p}^+ = 2 \frac{f}{c} D_0 \boldsymbol{n}$$

Warning: $\boldsymbol{p}$ is a row vector like $\boldsymbol{n}^+$.

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{n}}} \cdot \dot{\boldsymbol{n}} + \dot{\boldsymbol{n}}^+ \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{n}}^+} = 2 f \left(D^{0\,*} \boldsymbol{n}^+ \partial_0 \boldsymbol{n} + \partial^0 \boldsymbol{n}^+ D_0 \boldsymbol{n} \right)$$

$$= 2 f \left[D^{0\,*} \boldsymbol{n}^+ D_0 \boldsymbol{n} + i K_0 D^{0\,*} \boldsymbol{n}^+ \boldsymbol{n} + \partial^0 \boldsymbol{n}^+ D_0 \boldsymbol{n} \right) \right]$$

$$= 2 f \left[D^{0\,*} \boldsymbol{n}^+ D_0 \boldsymbol{n} + i K_0 \left(\partial^0 - i K^0 \right) \boldsymbol{n}^+ \boldsymbol{n} + \partial^0 \boldsymbol{n}^+ (\partial_0 + i K_0) \boldsymbol{n} \right) \right]$$

$$= 2 f \left(D^{0\,*} \boldsymbol{n}^+ D_0 \boldsymbol{n} + \frac{1}{c^2} \dot{\boldsymbol{n}}^+ \dot{\boldsymbol{n}} - K_0^2 \right) \qquad (\boldsymbol{n}^+ \boldsymbol{n} = 1)$$

$$\therefore \qquad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{n}}} \cdot \dot{\boldsymbol{n}} + \dot{\boldsymbol{n}}^+ \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{n}}^+} - \mathcal{L}$$

$$= 2 f \left(\frac{1}{c^2} \dot{\boldsymbol{n}}^+ \dot{\boldsymbol{n}} - K_0^2 + D_i^* \boldsymbol{n}^+ D_i \boldsymbol{n} \right)$$

For a static “magnetic” field,

$$\dot{\boldsymbol{n}} = \dot{\boldsymbol{n}}^+ = 0 \qquad\qquad K_0 = 0$$

$$\rightarrow \qquad \mathcal{H} = 2 f D_i^* \boldsymbol{n}^+ D_i \boldsymbol{n} = 2 f (D_i \boldsymbol{n})^+ D_i \boldsymbol{n} = 2 f (\boldsymbol{D} \boldsymbol{n})^+ \cdot (\boldsymbol{D} \boldsymbol{n})$$

Reminder: $\boldsymbol{D}_i \equiv D^i = -D_i = -(\partial_i + i K_i) = -(\nabla - i \boldsymbol{K})_i$

$$= \partial^i + i K^i = (-\nabla + i \boldsymbol{K})_i$$

In a plane,

$$H = 2 f \int d^2 r \, (D_i \boldsymbol{n})^+ D_i \boldsymbol{n}$$

$$= 2 f \int d^2 r \, (\boldsymbol{D} \boldsymbol{n})^+ \cdot (\boldsymbol{D} \boldsymbol{n})$$