7.8.c. Projective Representation

CP^{N-1} is just the $(N-1)$-D complex projective space.
[See A.d'Adda et al., Nucl.Phys. B146, 63 (1978)]

Writing

$$\boldsymbol{n}(x) = \Lambda(x)\,(\,\omega_1(x),\;\ldots,\;\omega_N(x)\,)$$

then

$$\boldsymbol{n}^{+}\,\boldsymbol{n} = 1 \qquad \to \qquad \Lambda^{-2} = \omega_a^{*}\,\omega_a$$

In a domain where $\omega_a(x) \neq 0$, we can write

$$\boldsymbol{n}(x) = \omega_a(x)\left(\,\eta_1^{(a)}(x),\;\ldots,\;\eta_N^{(a)}(x)\right)$$

where

$$\eta_b^{(a)} = \frac{\omega_b(x)}{\omega_a(x)} \qquad \text{so that } \eta_a^{(a)} = 1$$

Let $\Lambda^{(a)}(x) = |\,\omega_a(x)\,|$ then

$$\boldsymbol{n}(x) = e^{i\,\alpha(x)}\,\Lambda^{(a)}(x)\left(\,\eta_1^{(a)}(x),\;\ldots,\;\eta_N^{(a)}(x)\right)$$

where

$$e^{i\,\alpha(x)} = \frac{\omega_a(x)}{|\,\omega_a(x)\,|}$$

is the phase of $\omega_a(x)$.

Since the Langrangian is invariant under a local gauge transformation, $e^{i\,\alpha(x)}$ can be removed by adjusting the gauge field K_μ. Hence, we can replace $\boldsymbol{n}$ with

$$\boldsymbol{n}'(x) = \Lambda^{(a)}(x)\left(\,\eta_1^{(a)}(x),\;\ldots,\;\eta_N^{(a)}(x)\right)$$

which has only $N-1$ independent components, while $\Lambda^{(a)}(x)$ is real & positive definite everywhere.

$\boldsymbol{n}'(x)$ is called the projective representation.