## 7.8.d. Topological Charge

Analogous to the vortex case discussed in 7.5.a._FluxQuantization.pdf, we set

$$J_s^\mu = -\alpha\, \varepsilon^{\mu\nu\lambda 3}\, \partial_\nu K_\lambda \qquad (\alpha > 0)$$

& $$Q_s = \int d^2 r\, J_s^0 = -\alpha \int d^2 r\, \varepsilon^{0jk3}\, \partial_j K_k$$

Using $\boldsymbol{n}^+ \boldsymbol{n} = 1$, we have

$$\begin{aligned}(D_j\, \boldsymbol{n})^+ (D_k\, \boldsymbol{n}) &= (\partial_j - i K_j)\, \boldsymbol{n}^+ (\partial_k + i K_k)\, \boldsymbol{n} \\ &= \partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n} + K_j K_k - i\left(K_j\, \boldsymbol{n}^+ \partial_k \boldsymbol{n} - K_k\, \partial_j \boldsymbol{n}^+ \boldsymbol{n}\right) \\ &= \partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n} + K_j K_k - i\left(K_j\, \boldsymbol{n}^+ \partial_k \boldsymbol{n} + K_k\, \boldsymbol{n}^+ \partial_j \boldsymbol{n}\right)\end{aligned}$$

where we've used

$$\partial_j \boldsymbol{n}^+ \boldsymbol{n} = \partial_j (\boldsymbol{n}^+ \boldsymbol{n}) - \boldsymbol{n}^+ \partial_j \boldsymbol{n} = -\boldsymbol{n}^+ \partial_j \boldsymbol{n}$$

Since all terms invariant under $j \leftrightarrow k$ vanish if multiplied by $\varepsilon_{0jk3}$,

$$\varepsilon_{0jk3} (D_j\, \boldsymbol{n})^+ (D_k\, \boldsymbol{n}) = \varepsilon_{0jk3}\, \partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n}$$

From 7.8.b._CP(N-1)SigmaModel.pdf, we have

$$K^\mu = i\, \boldsymbol{n}^+ \partial^\mu \boldsymbol{n}$$

→ $$\partial_j K_k = i\, \partial_j (\boldsymbol{n}^+ \partial_k \boldsymbol{n}) = i\left(\partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n} + \boldsymbol{n}^+ \partial_j \partial_k \boldsymbol{n}\right)$$

$$\varepsilon^{0jk3}\, \partial_j K_k = i\, \varepsilon_{0jk3}\, \partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n}$$

∴ $$\begin{aligned}Q_s &= -i\,\alpha \int d^2 r\, \varepsilon_{0jk3}\, \partial_j \boldsymbol{n}^+ \partial_k \boldsymbol{n} \\ &= -i\,\alpha \int d^2 r\, \varepsilon_{0jk3} (D_j\, \boldsymbol{n})^+ (D_k\, \boldsymbol{n})\end{aligned}$$