7.8.e. Skyrmion Solutions

Starting with (c.f. 7.7.b._GeneralSolutionsForSkyrmions.pdf)

$$\boldsymbol{m}_i^{(\pm)} = D_i\, \boldsymbol{n} \pm i\, \varepsilon_{0ij3}\, D_j\, \boldsymbol{n}$$

$$\boldsymbol{m}_i^{(\pm)+} = D_i^*\, \boldsymbol{n}^+ \mp i\, \varepsilon_{0ij3}\, D_j^*\, \boldsymbol{n}^+$$

we have

$$\boldsymbol{m}_i^{(\pm)+}\, \boldsymbol{m}_{\underline{i}}^{(\pm)} \geq 0 \quad \rightarrow \quad \boldsymbol{m}_i^{(\pm)+}\, \boldsymbol{m}_i^{(\pm)} \geq 0$$

i.e., $\left(D_i^*\, \boldsymbol{n}^+ \mp i\, \varepsilon_{0ij3}\, D_j^*\, \boldsymbol{n}^+\right)\left(D_i\, \boldsymbol{n} \pm i\, \varepsilon_{0ik3}\, D_k\, \boldsymbol{n}\right) \geq 0$

Considering the 4 terms in the product separately, we have

$$D_i^*\, \boldsymbol{n}^+\, D_i\, \boldsymbol{n} = (D_i\, \boldsymbol{n})^+\, D_i\, \boldsymbol{n}$$

$$\varepsilon_{0ij3}\, D_j^*\, \boldsymbol{n}^+\, D_i\, \boldsymbol{n} = -\varepsilon_{0ik3}\, D_i^*\, \boldsymbol{n}^+\, D_k\, \boldsymbol{n}$$

$$\varepsilon_{0ij3}\, D_j^*\, \boldsymbol{n}^+\, \varepsilon_{0ik3}\, D_k\, \boldsymbol{n} = \delta_{jk}\, D_j^*\, \boldsymbol{n}^+\, D_k\, \boldsymbol{n} \qquad (j, k = 1, 2)$$

$$= D_j^*\, \boldsymbol{n}^+\, D_j\, \boldsymbol{n}$$

$\therefore \quad (D_i\, \boldsymbol{n})^+\, D_i\, \boldsymbol{n} \pm i\, \varepsilon_{0ij3}\, D_i^*\, \boldsymbol{n}^+\, D_j\, \boldsymbol{n} \geq 0$

$\rightarrow \quad \int d^2 r\, (D_i\, \boldsymbol{n})^+\, D_i\, \boldsymbol{n} \geq \mp i \int d^2 r\, \varepsilon_{0ij3} (D_i\, \boldsymbol{n})^+\, D_j\, \boldsymbol{n}$

With $\boldsymbol{D} = (D_1, D_2)$ understood, we can write

$$\int d^2 r\, (\boldsymbol{D}\, \boldsymbol{n})^+ \cdot (\boldsymbol{D}\, \boldsymbol{n}) \geq \mp i \int d^2 r\, \left[(\boldsymbol{D}\, \boldsymbol{n})^+ \times (\boldsymbol{D}\, \boldsymbol{n})\right] \cdot \hat{\boldsymbol{z}}$$

From 7.8.b._CP(N-1)SigmaModel.pdf, we see that in a plane,

$$H = 2f \int d^2 r\, (D_i\, \boldsymbol{n})^+\, D_i\, \boldsymbol{n} = 2f \int d^2 r\, (\boldsymbol{D}\, \boldsymbol{n})^+ \cdot (\boldsymbol{D}\, \boldsymbol{n})$$

From 7.8.d._TopologicalCharge.pdf, we have

$$Q_s = -i\, \alpha \int d^2 r\, \varepsilon_{0jk3}\, (D_j\, \boldsymbol{n})^+\, (D_k\, \boldsymbol{n})$$

Hence,

$$H \geq \pm \frac{2f}{\alpha}\, Q_s > 0$$

Note: The overall signs in the definitions of H & Q_s were carefully chosen so that the correct end results (see 7.8.g._Example.pdf) require f & α to be real & positive.

Skyrmions are at the energy minima

$$H = \pm \frac{2f}{\alpha}\, Q_s$$

i.e.,

$$D_i\, \boldsymbol{n} \pm i\, \varepsilon_{0ij3}\, D_j\, \boldsymbol{n} = 0$$

or $\quad D_i\, n_a = \mp i\, \varepsilon_{0ij3}\, D_j\, n_a \qquad \forall\ a = 1, \ldots, N$

Set $\quad n_a = n_A\, \omega_a \qquad$ where $A = 1, \ldots,$ or, N

$\rightarrow \quad D_i\, n_a = (\partial_i - i\, K_i)\, (n_A\, \omega_a)$

$$= \partial_i n_A\, \omega_a + n_A\, \partial_i \omega_a - i\, K_i\, n_A\, \omega_a$$

$$= D_i\, n_A\, \omega_a + n_A\, \partial_i \omega_a$$

$$= \mp i\, \varepsilon_{0ij3}\, D_j\, n_a$$

$$= \mp i\, \varepsilon_{0ij3} \left(D_j\, n_A\, \omega_a + n_A\, \partial_j \omega_a\right)$$

$$D_i\, n_A = \mp i\, \varepsilon_{0ij3}\, D_j\, n_A$$

→ $$n_A\, \partial_i\, \omega_a = \mp i\, \varepsilon_{0ij3}\, n_A\, \partial_j\, \omega_a$$

$$\partial_i\, \omega_a = \mp i\, \varepsilon_{0ij3}\, \partial_j\, \omega_a$$

i.e., $\partial_x\, \omega_a = \mp i\, \partial_y\, \omega_a$

As in 7.7.b._GeneralSolutionsForSkyrmions.pdf, let

$$z = x + i\, y \qquad\qquad z^* = x - i\, y$$

→ $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\, \frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\, \frac{\partial}{\partial y}\right)$$

Taking the upper sign, which corresponds to $Q_s > 0$ & skyrmions, we have

$$\frac{\partial\, \omega_a}{\partial z^*} = \frac{1}{2}\left(-i\, \partial_y\, \omega_a + i\, \partial_y\, \omega_a\right) = 0$$

i.e., $\omega_a = \omega_a(z)$

Taking the lower sign, which corresponds to $Q_s < 0$ & anti-skyrmions, we have

$$\frac{\partial\, \omega_a}{\partial z} = \frac{1}{2}\left(i\, \partial_y\, \omega_a - i\, \partial_y\, \omega_a\right) = 0$$

i.e., $\omega_a = \omega_a\,(z^*)$