7.8.f. Topological Charge Density

In the projective representation (see 7.8.c._ProjectiveRepresentative.pdf),

$$\boldsymbol{\omega} = \Lambda^{-1}\, \boldsymbol{n}$$

where Λ is real & positive definite.

For skyrmions,

$$\frac{\partial \omega_a}{\partial z^*} = 0$$

$$\rightarrow \qquad \Lambda^{-1} \frac{\partial n_a}{\partial z^*} + \frac{\partial \Lambda^{-1}}{\partial z^*} n_a = 0$$

$\Lambda\, n_a^*$ times both sides gives

$$n_a^* \frac{\partial n_a}{\partial z^*} + \Lambda \frac{\partial \Lambda^{-1}}{\partial z^*} = 0 \qquad \text{since} \qquad n_a^*\, n_a = \boldsymbol{n}^+ \boldsymbol{n} = 1$$

$$\rightarrow \qquad n_a^* \frac{\partial n_a}{\partial z^*} + \frac{\partial \ln \Lambda^{-1}}{\partial z^*} = 0$$

Using

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

we have

$$n_a^* \left(\partial_x n_a + i\, \partial_y n_a \right) + \partial_x \ln \Lambda^{-1} + i\, \partial_y \ln \Lambda^{-1} = 0$$

From 7.8.b._CP(N-1)SigmaModel.pdf,

$$K^\mu = i\, \boldsymbol{n}^+ \partial^\mu \boldsymbol{n}$$

$$\rightarrow \qquad K_\mu = i\, \boldsymbol{n}^+ \partial_\mu \boldsymbol{n}$$

$$K_i = i\, \boldsymbol{n}^+ \partial_i \boldsymbol{n} \;= -\boldsymbol{K}_i \qquad \text{i.e.,} \qquad \boldsymbol{K} = -i\, \boldsymbol{n}^+ \nabla \boldsymbol{n}$$

$$\therefore \qquad i\, \boldsymbol{K}_x - \boldsymbol{K}_y + \partial_x \ln \Lambda^{-1} + i\, \partial_y \ln \Lambda^{-1} = 0$$

Since K^μ & Λ are real, we have

$$\boldsymbol{K}_x = -\partial_y \ln \Lambda^{-1}$$

$$\boldsymbol{K}_y = \partial_x \ln \Lambda^{-1}$$

i.e.,

$$\varepsilon_{0 i j 3}\, \boldsymbol{K}_j = \partial_i \ln \Lambda^{-1}$$

$$\boldsymbol{\omega} = \Lambda^{-1}\, \boldsymbol{n} \qquad\qquad \boldsymbol{n}^+ \boldsymbol{n} = 1$$

$$\rightarrow \qquad \Lambda^{-1} = \boldsymbol{n}^+ \boldsymbol{\omega} \;= \Lambda\, \boldsymbol{\omega}^+ \boldsymbol{\omega}$$

$$= \sqrt{\boldsymbol{\omega}^+ \boldsymbol{\omega}} \;= \sqrt{\omega_a^*\, \omega_a}$$

$$\therefore \qquad \varepsilon_{0 i j 3}\, \boldsymbol{K}_j = \partial_i \ln \sqrt{\omega_a^*\, \omega_a}$$

$$\varepsilon_{0 i k 3}\, \varepsilon_{0 i j 3}\, \boldsymbol{K}_j = \delta_{k j}\, \boldsymbol{K}_j \;= \boldsymbol{K}_k \;= \varepsilon_{0 i k 3}\, \partial_i \ln \sqrt{\omega_a^*\, \omega_a}$$

$$\rightarrow \qquad \boldsymbol{K}_i = -\varepsilon_{0 i k 3}\, \partial_k \ln \sqrt{\omega_a^*\, \omega_a}$$

From 7.8.d._TopologicalCharge.pdf, the topological charge density is

$$J_s^0 = -\alpha\, \varepsilon_{0 j k 3}\, \partial_j K_k$$

$$= \alpha\, \varepsilon_{0 j k 3}\, \partial_j \boldsymbol{K}_k$$

$$= \alpha\, \varepsilon_{0 j k 3}\, \varepsilon_{0 k i 3}\, \partial_j \partial_i \ln \sqrt{\omega_a^*\, \omega_a}$$

$$= -\alpha\, \delta_{j\,i}\, \partial_j \partial_i \ln \sqrt{\omega_a^* \,\omega_a} \qquad (\, i,\ j = 1,\, 2\,)$$

$$= -\alpha\, \partial_i \partial_i \ln \sqrt{\omega_a^* \,\omega_a}$$

$$= -\alpha\, \nabla^2 \ln \sqrt{\omega_a^* \,\omega_a} \qquad (\, \nabla^2 \text{ here is 2-D}\,)$$

$$Q_s = \int d^2 r\, J_s^0 = \alpha \int d^2 r\ \varepsilon_{0\,j\,k\,3}\, \partial_j \boldsymbol{K}_k$$

$$= \alpha \int_S d\, \boldsymbol{S} \cdot (\nabla \times \boldsymbol{K}) \qquad (\, S = \text{ xy-plane}\,)$$

$$= \alpha \oint_C d\,\boldsymbol{r} \cdot \boldsymbol{K} \qquad (\, C = \text{boundary of } S\,)$$

$$= -i\, \alpha \oint_C d\,\boldsymbol{r} \cdot \left(\boldsymbol{n}^+ \nabla \boldsymbol{n} \right)$$