7.8.g. Example

A simple but nontrivial example is

$$\boldsymbol{\omega}(z) = z\,\boldsymbol{a} + \kappa\,\boldsymbol{b}$$

where κ is real while $\boldsymbol{a}$ & $\boldsymbol{b}$ are orthonormal constant vectors, i.e.,

$$\boldsymbol{a}^{+}\boldsymbol{a} = \boldsymbol{b}^{+}\boldsymbol{b} = 1 \qquad \boldsymbol{a}^{+}\boldsymbol{b} = 0$$

$$\boldsymbol{\omega}^{+} = z^{*}\boldsymbol{a}^{+} + \kappa\,\boldsymbol{b}^{+}$$

→ $$\boldsymbol{\omega}^{+}\boldsymbol{\omega} = \left(z^{*}a^{+} + \kappa\,\boldsymbol{b}^{+}\right)(z\,\boldsymbol{a} + \kappa\,\boldsymbol{b})$$

$$= |\,z\,|^{2} + \kappa^{2}$$

$$\boldsymbol{n} = \frac{\boldsymbol{\omega}}{|\,\boldsymbol{\omega}\,|} = \frac{z\,\boldsymbol{a} + \kappa\,\boldsymbol{b}}{\sqrt{|\,z\,|^{2} + \kappa}}$$

For $r \to \infty$,

$$\boldsymbol{n}(x) = \boldsymbol{n}(t, \boldsymbol{r}) \quad \to \quad \frac{z\,\boldsymbol{a}}{|\,z\,|} = e^{i\theta}\boldsymbol{a}$$

Thus, the phase of $\boldsymbol{n}$ increases by 2π going around the boundary.
We expect it to be a skyrmion of order 1, i.e., $Q_s = +1$.

Reminder:

$$z = x + i\,y \qquad |\,z\,| = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

From 7.8.f._TopologicalChargeDensity.pdf,

$$Q_s = -i\,\alpha \oint_C d\boldsymbol{r}\cdot\left(\boldsymbol{n}^{+}\nabla\boldsymbol{n}\right)$$

Setting C = circle of radius R on the xy-plane, we have

$$d\boldsymbol{r} = \hat{\boldsymbol{\theta}}\,R\,d\theta \qquad d\boldsymbol{r}\cdot\nabla = R\,d\theta\,\frac{\partial}{R\,\partial\theta} = d\theta\,\frac{\partial}{\partial\theta}$$

∴ $$Q_s = -i\,\alpha \int_0^{2\pi} d\theta\,\left(e^{-i\theta}\boldsymbol{a}^{+}\right)\frac{\partial}{\partial\theta}\left(e^{i\theta}\boldsymbol{a}\right)$$

$$= 2\pi\alpha$$

$$= 1$$

→ $$\alpha = \frac{1}{2\pi}$$