8.1. Spin & Statistics

Ref: A.Khare,"Fractional Statistics & Quantum Theory", 1997, Chap.2.

Anyons = Particles obeying fractional statistics.

Particle statistics is determined by the phase factor $e^{i\alpha}$ picked up by the wave function under the interchange of the positions of any pair of (identical) particles in the system.

Before the discovery of the anyons, this particle interchange (or exchange) was treated as the permutation of particle labels. Let P be the operator for this interchange.

$$P^2 = I \quad \rightarrow \quad e^{2i\alpha} = 1$$

$$\therefore \quad e^{i\alpha} = \pm 1 \quad \text{i.e.,} \quad \alpha = 0,\ \pi$$

Thus, there're only 2 kinds of statistics, $\alpha = 0\,(\pi)$ for Bosons (Fermions) obeying Bose-Einstein (Fermi-Dirac) statistics.

Pauli's spin-statistics theorem then relates particle spin with statistics, namely, bosons (fermions) are particles with integer (half-integer) spin.

To account for the anyons, particle exchange is re-defined as an observable adiabatic (constant energy) process of physically interchanging particles. (This is in line with the quantum philosophy that only observables are physically relevant.)

As will be shown later, the new definition does not affect statistics in 3-D space. However, for particles in 2-D space, α can be any (real) value; hence anyons.

The converse of the spin-statistic theorem then implies arbitrary spin for 2-D particles.

Quantization of *S* in 3-D

See M.Alonso, H.Valk, "Quantum Mechanics: Principles & Applications", 1973, §6.2.

In 3-D, the (spin) angular momentum ***S*** has 3 non-commuting components satisfying

$$\left[S_i,\ S_j \right] = i\hbar\,\varepsilon_{ijk}\,S_k$$

$$\& \quad \left[\mathbf{S}^2,\ S_i \right] = 0$$

This means a state can be the simultaneous eigenstate of $\mathbf{S}^2$ & at most one S_i. The common practice is to choose $\mathbf{S}^2$ & S_3 so that

$$\mathbf{S}^2 \left| \eta m \right\rangle = \eta\hbar^2 \left| \eta m \right\rangle \qquad S_3 \left| \eta m \right\rangle = m\hbar \left| \eta m \right\rangle$$

where η & m are the respective (dimensionless) eigenvalues.

Define the ladder operators

$$S_\pm = S_1 \pm i\,S_2$$

With a bit of tedious but straightforward calculation, we have

$$S_\pm S_\mp = S_1^2 + S_2^2 \pm \hbar S_3$$

$$= \mathbf{S}^2 - S_3(S_3 \mp \hbar)$$

$$S^2 = \frac{1}{2}(S_+ S_- + S_- S_+) + S_3^2$$

$$\& \quad \left[\mathbf{S}^2,\ S_\pm \right] = 0$$

$$[S_3,\ S_\pm] = \pm\hbar S_\pm$$

$$[S_\pm,\ S_\mp] = \pm 2\hbar S_3$$

Now,

$$[S_3,\ S_\pm] = \pm\hbar S_\pm \quad \rightarrow \quad S_3 S_\pm = S_\pm(S_3 \pm \hbar)$$

$$\therefore \quad S_3 S_\pm \left| \eta m \right\rangle = (m \pm 1)\hbar S_\pm \left| \eta m \right\rangle$$

i.e., $S_\pm \mid \eta\, m \rangle = \alpha_\pm \mid \eta\, m \pm 1 \rangle$ ($\alpha_\pm$ = constant)

Thus, $S_\pm$ raises/lower m by 1.
Since the value of S_3 must be finite, we have

$$S_+ \mid \eta\, m_{max} \rangle = 0 \qquad\qquad S_- \mid \eta\, m_{min} \rangle = 0$$

$$\to \quad S_- S_+ \mid \eta\, m_{max} \rangle = 0 \qquad\qquad S_+ S_- \mid \eta\, m_{min} \rangle = 0$$

$$\left[\boldsymbol{S}^2 - S_3 (S_3 + \hbar) \right] \left| \eta\, m_{max} \right\rangle = 0 \qquad \left[\boldsymbol{S}^2 - S_3 (S_3 - \hbar) \right] \left| \eta\, m_{min} \right\rangle = 0$$

$$\therefore \quad \eta - m_{max}(m_{max} + 1) = 0 \qquad \eta - m_{min}(m_{min} - 1) = 0$$

$$\to \quad m_{max}(m_{max} + 1) = m_{min}(m_{min} - 1)$$

$$(m_{max} + m_{min})\,(m_{max} - m_{min} + 1) = 0$$

i.e., $m_{max} = -m_{min}$ or $m_{max} = m_{min} - 1$

Since $m_{max} \geq m_{min}$, we can only have

$$m_{max} = -m_{min}$$

Starting from $m = m_{min}$, we can use S_+ for $m_{max} - m_{min} = 2\, m_{max}$ times to reach m_{max}. Thus, $2\, m_{max}$ must be an integer. By convention, the spin s of the particle is given by $s = m_{max}$. Hence

$$s = \frac{1}{2} n \text{ with } \quad n = 0, 1, 2, \ldots$$

i.e., $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots.$

Thus,

$$\eta = s(s+1)$$

but it’s conventional to denote the eigenstates as $\mid s\, m \rangle$ so that

$$\boldsymbol{S}^2 \left| s\, m \right\rangle = s(s+1)\, \hbar^2 \left| s\, m \right\rangle \qquad S_3 \mid s\, m \rangle = m\, \hbar \mid s\, m \rangle$$

Any S in 2-D

In 2-D, $\boldsymbol{S}$ has only 1 component, say, $\boldsymbol{S} = S_3\, \hat{\boldsymbol{z}}$.
Since

$$\left[\boldsymbol{S}^2, S_3 \right] = \left[S_3^2, S_3 \right] = 0$$

The simultaneous eigenstates require only one label

$$\boldsymbol{S}^2 \left| m \right\rangle = m^2\, \hbar^2 \left| m \right\rangle \qquad S_3 \mid m \rangle = m\, \hbar \mid m \rangle$$

The finiteness of m , i.e.,

$$s \geq m \geq -s$$

brings in no restriction on the possible values of s, as advertised.

Particle Exchange

See Khare, §2.3.

Let the configuration space of 1 particle be $\mathcal{X}$. If the particles are identical (i.e., they have the same physical attributes such as mass, charge, etc.) but distinguishable (i.e., we can tell which is which after they collide), the configuration space of N particles is simply the direct product space $\mathcal{X}^N$.

However, if the particles are indistinguishable (as do all identical particles in quantum mechanics), the set of all points in $\mathcal{X}^N$ that are related by particle exchanges must be treated as a single point. Thus, the configuration space becomes $\mathcal{X}^N / \mathbb{S}_N$, where $\mathbb{S}_N$ is the group for permutating N objects.

For example, the Gibbs paradox in classical statistical mechanics was resolved by shrinking the volume of $\mathcal{X}^N$ by a factor of $(N!)^{-1}$.

The easiest way to study the effect of particle exchange is in terms of relative coordinates

$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ so that the exchange $i \leftrightarrow j$ simply means $\boldsymbol{r}_{ij} \to \boldsymbol{r}_{ji} = -\boldsymbol{r}_{ij}$.

However, each point $\boldsymbol{r}_{ij} = 0$ represents a singularity since one cannot determine whether the particles were exchanged or not because $\boldsymbol{r}_{ij} = 0 = -0 = \boldsymbol{r}_{ji}$. The accessible configuration space must therefore exclude these singularities.

Mathematically, this can be denoted by writing

$$\mathcal{X}^N / \mathbb{S}_N = R^d \otimes r(d, N)$$

where d is the spatial dimension of the system, R^d the Euclidean space for the center of mass motion, & $r(d, N)$ is the $d(N-1)$-D, exchange-identified, singularities excluded, configuration space in relative coordinates.

The effects of particle exchange can then be classified according to the topologically distinct closed paths on $r(d, N)$. Which means the phase factor $e^{i\alpha}$ is just the 1-D representation of the fundamental homotopy group

$$\pi_1[r(d, N)] = \pi_1\left(\mathcal{X}^N / \mathbb{S}_N\right) \qquad [\, \pi_1\left(R^d\right) = 0 \,]$$

Example: $N = 2$, $d = 3$

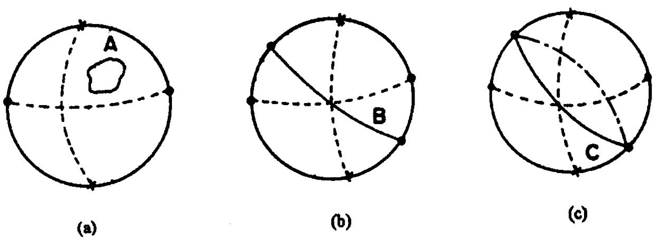
For $N = 2$, we have

$$r(d, 2) = \frac{R^d - \{0\}}{\mathbb{S}_2} = (0, \infty) \otimes \mathrm{RP}^{d-1}$$

where $(0, \infty)$ is the positive real line with the origin omitted, & RP^{d-1} is the real projective space obtained from $R^d - \{0\}$ by 1st making every point $\boldsymbol{r}$ equivalent to $\hat{\boldsymbol{r}}$ & then identifying every $\hat{\boldsymbol{r}}$ with $-\hat{\boldsymbol{r}}$.

For $d = 3$, $\mathrm{RP}^{d-1} = \mathrm{RP}^2$ is the unit sphere S^2 with the points at both ends of each diameter identified. Thus, $r(3, 2)$ is the solid sphere of infinite radius minus the center. However, in considering the effects of exchange, all closed paths can be continuously contracted to one on a sphere of radius $|\boldsymbol{r}_{ij}|$, which in turn is equivalent to S^2.

We shall begin by considering the closed paths shown in the figure.



For path ***A*** in fig.(a), the particle starting at any $\boldsymbol{r} = \boldsymbol{r}_0$ simply returns to $\boldsymbol{r}_0$. Since $\boldsymbol{r}$ is never near $-\boldsymbol{r}_0$, there is no exchange. The defining topological characteristic of path ***A*** is that it can be shrinked continuously to a point.

Path ***B*** in fig.(b) starts at a point $\boldsymbol{r}_0$ on one end of a diameter & ends up at point $-\boldsymbol{r}_0$ on the other end of the diameter. Thus, there is a single exchange. We can call the path a "loop" because the starting & ending points are identified.

Note that path ***B*** cannot be shrinked continuously to a point since, in order to remain a loop, its end points must be at the opposite ends of a diameter. This is also the defining topological characteristic of $\alpha = \pi$ loops.

For path ***C*** in fig.(c), the particle starts at one end of a diameter, reaches the other end, then returns to the starting point. Thus, there are two exchanges. However, can be shrinked continuously to a point so that it is topologically equivalent to path ***A***.

Further considerations easily show that there are only two types, ***A*** & ***B***, of topologically distinct closed paths.

In terms of homotopy theory,

$$\pi_1(\mathrm{RP}^2) = \mathbb{S}_2$$

There are only two 1-D (irreducible) representations in $\mathbb{S}_2$ given by

$\mathbb{S}_2$	e	(12)
Γ_S	1	1
Γ_A	1	−1

where, for our purposes,

e = even number of exchanges (transport along type ***A*** paths)

(12) = odd number of exchanges (transport along type ***B*** paths)

Obviously, Γ_S gives $e^{i\alpha}$ for bosons & Γ_A fermions.

Example: $N = 2$, $d = 2$

For $d = 2$, $\mathrm{RP}^{d-1} = \mathrm{RP}^1$ is the unit circle S^1 with the points at θ & $\theta + \pi$ identified.
This can be done by cutting the circle & wrap it up so that the point opposite the cut point is brought adjacent to it (see fig.a).

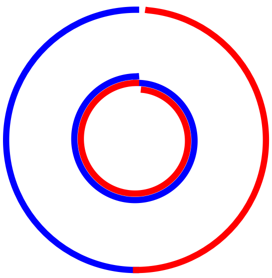


fig.a

The outer circle S^1 is cut & winded up to get the inner figure RP^1 so that every point on the red arc is now adjacent to its counterpart on the blue arc.

Note winding once ($\Delta\theta = 2\pi$) on RP^1 corresponds to going half-way on S^1 with $\Delta\theta = \pi$.

Thus, $r(2, 2)$ is a disk of infinite radius minus the center & with every point (r, θ) & $(r, \theta + \pi)$ identified. Repeating the above folding action, we see that $r(2, 2)$ is a cone of half-angle 30°, infinite height, & minus its tip, as shown in fig.b.

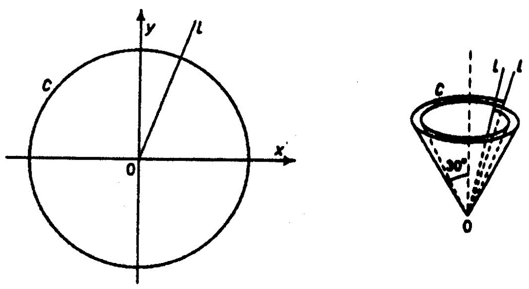


fig.b

Analogous to the $d = 3$ case, we need only consider closed paths on S^1 with the points at θ & $\theta + \pi$ identified.

The fundamental group is

$$\pi_1(\mathrm{RP}^1) = \mathbb{Z}$$

There are countably infinite number of 1-D representations for $\mathbb{Z}$.

$\mathbb{Z}$	...	−2	−1	0	1	2	...	m	...
Γ_0	...	1	1	1	1	1	...	1	...
Γ_1	...	$e^{-2i\pi}$	$e^{-i\pi}$	1	$e^{i\pi}$	$e^{2i\pi}$	...	$e^{im\pi}$	...
Γ_{-1}	...	$e^{2i\pi}$	$e^{i\pi}$	1	$e^{-i\pi}$	$e^{-2i\pi}$	...	$e^{-im\pi}$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Γ_n		$e^{-2i\pi/n}$	$e^{-i\pi/n}$	1	$e^{i\pi/n}$	$e^{2i\pi/n}$		$e^{im\pi/n}$	

where, for our purposes,

m = number of windings on RP^1

$|\, m \,|$ = number of exchanges

Winding is signed since the particles cannot go pass each other.

Thus α can be any rational number.

General Case

We quote without proof that (see Khare)

$$\pi_1\left(X^N / \mathbb{S}_N\right) = \begin{cases} B_N & \text{for } d = 2 \\ \mathbb{S}_N & \text{for } d \geq 3 \end{cases}$$

where B_N is the braid group of N objects.