8.3. Quantum Mechanics

In Feynman's path integral formulism of quantum mechanics, the fundamental quantity is the transition amplitude given by

$$P = \langle t_1, \boldsymbol{r}_1 \mid t_0, \boldsymbol{r}_0 \rangle = C \sum_{\text{all paths}} e^{i S / \hbar}$$

where C is a normalization constant & S is the action

$$S = \int_{t_0}^{t_1} d t \, L$$

The assumption here is that the contribution of a path is weighted only by its dynamical factor $e^{i S / \hbar}$, thus implying that all paths are geometrically (topologically) equivalent. Obviously, the assumption holds only when the configuration space of the system is simply connected.

For the general case, the formula should be modified to read

$$P = C \sum_{\eta} \chi(\eta) \sum_{\text{path}_\eta} e^{i S / \hbar}$$

where path_η denotes paths belonging to the homotopy class η & $\chi(\eta)$ the associated weighing factor.

The simplest choice is to set $\chi(\eta)$ to be an 1-D representation of π_1. Hence,

$$\chi(\eta_1) \, \chi(\eta_2) = \chi(\eta_1 \, \eta_2)$$

The general solution is

$$\chi(\eta) = e^{\pm i \pi \phi_\eta}$$

where $0 \le \phi_\eta < 1$.

For a system of 2 particles in a plane, such a factor can be introduced by adding to L on path_α an "interacting" term

$$L_S(\alpha) = \hbar \, \alpha \frac{d \theta_{12}}{d t} = \hbar \, \alpha \frac{d \theta_{21}}{d t}$$

where $\theta_{a b}$ is the azimuthal angle of the vector $\boldsymbol{r}_{a b} \equiv \boldsymbol{r}_b - \boldsymbol{r}_a$ & $\theta_{b a} = \theta_{a b} + \pi$.

Note: $\theta(\boldsymbol{x}_r - \boldsymbol{x}_s)_{\text{Ezawa}} = \theta_{s r}$

Thus, for closed paths,

$$\frac{i}{\hbar} \int_{t_0}^{t_1} d t \, L_S(\alpha) = i \, \alpha \, [\, \theta_{12}(t_1) - \theta_{12}(t_0) \,]$$

$$= i \, n \, \pi \, \alpha \qquad\qquad n = 0, \pm 1, \pm 2, \ \ldots$$

Note: $\Delta \theta_{12} = \pm \pi$ (with $\Delta \, r_{12} = 0$ understood) denotes a particle exchange as well as a closed path.

$\therefore \qquad \chi(\eta) = e^{i n \pi \alpha} \qquad$ with $\qquad \pm \phi_\eta = n \, \alpha \qquad$ & $\qquad 0 \le \alpha < 1$

For convenience, we'll simply write

$$\chi(\alpha) = e^{i n \pi \alpha} \qquad 0 \le \alpha < 1$$

For N ordinary non-interacting particles in external potential V is

$$L_0 = \frac{1}{2} M \sum_{a=1}^{N} \left(\frac{d \, \boldsymbol{r}_a}{d t} \right)^2 - V(\boldsymbol{r}_1, \ \ldots, \boldsymbol{r}_N)$$

The corresponding Lagrangian for anyons is

$$L = L_0 + L_s$$

with

$$L_s = \frac{1}{2}\hbar\alpha \sum_{a \neq b}^{N} \frac{d\theta_{ab}}{dt} = \hbar\alpha \sum_{a<b}^{N} \frac{d\theta_{ab}}{dt}$$

Notation:

$$\boldsymbol{r} = (x_1, x_2) \qquad r^2 = x_1^2 + x_2^2 \qquad \theta = \tan^{-1}\frac{x_2}{x_1}$$

$$\boldsymbol{r}_a = (x_{a1}, x_{a2}) \qquad \boldsymbol{r}_{ab} = \boldsymbol{r}_b - \boldsymbol{r}_a$$

Also, rule of implicit summation over repeated indices does not apply to particle labels a, b, ...

$\sum_{a(\neq b)}$ denotes summation over all a that does not equal to b.

$$\sum_{a \neq b} \equiv \sum_a \sum_{b(\neq a)}$$

$$\frac{d\theta_{ab}}{dt} = \frac{d\theta(\boldsymbol{r}_{ab})}{dt} = \dot{\boldsymbol{r}} \cdot \nabla\theta(\boldsymbol{r}) \,|_{\boldsymbol{r}=\boldsymbol{r}_{ab}}$$
$$= \dot{x}_i \,\partial_i \theta(\boldsymbol{r}) \,|_{\boldsymbol{r}=\boldsymbol{r}_{ab}}$$

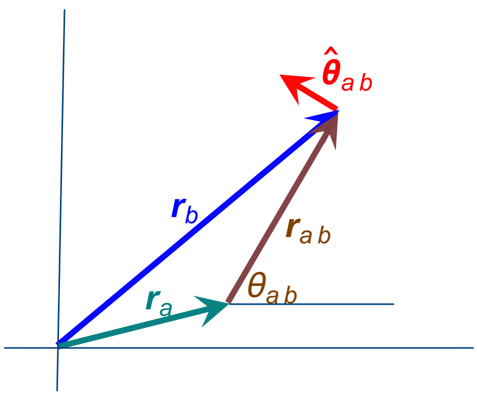
$$\partial_1\theta(\boldsymbol{r}) = -\frac{x_2}{r^2} \qquad \partial_2\theta(\boldsymbol{r}) = \frac{x_1}{r^2}$$

$$\rightarrow \quad \partial_i\theta(\boldsymbol{r}) = -\frac{1}{r^2}\varepsilon_{ij3}x_j \qquad \nabla\theta(\boldsymbol{r}) = -\frac{1}{r^2}\boldsymbol{r}\times\hat{z} = \frac{1}{r}\hat{\boldsymbol{\theta}}(\boldsymbol{r}) \qquad (\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}})$$

Thus,

$$\nabla_b\theta_{ab} = \frac{1}{r_{ab}}\hat{\boldsymbol{\theta}}_{ab} \qquad \nabla_a\theta_{ab} = -\frac{1}{r_{ab}}\hat{\boldsymbol{\theta}}_{ab}$$

where $\hat{\boldsymbol{\theta}}_{ab} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}_{ab} = -\hat{\boldsymbol{\theta}}_{ba}$



At first glance,

$$\nabla_b\theta_{ab} = -\nabla_a\theta_{ab}$$

may seem at odds with

$$\nabla_b\theta_{ab} = \nabla_b\theta_{ba} \quad (\theta_{ba} = \theta_{ab} + \pi)$$

However, there is no inconsistency. For example,

$$\nabla_b\theta_{ab} = \frac{1}{r_{ab}}\hat{\boldsymbol{\theta}}_{ab} = -\frac{1}{r_{ba}}\hat{\boldsymbol{\theta}}_{ba} = \nabla_b\theta_{ba}$$

$$\frac{\partial}{\partial\dot{\boldsymbol{r}}}\frac{d\theta(\boldsymbol{r})}{dt} = \frac{\partial}{\partial\dot{\boldsymbol{r}}}\left[\dot{\boldsymbol{r}}\cdot\nabla\theta(\boldsymbol{r})\right] = \nabla\theta(\boldsymbol{r}) = \frac{1}{r}\hat{\boldsymbol{\theta}}(\boldsymbol{r})$$

$$\rightarrow \qquad \frac{\partial}{\partial \dot{\boldsymbol{r}}_c} \frac{d\,\theta_{a\,b}}{d\,t} = (\delta_{c\,a}\,\nabla_a + \delta_{c\,b}\,\nabla_b)\,\theta_{a\,b} = (-\delta_{c\,a} + \delta_{c\,b})\,\frac{1}{r_{a\,b}}\,\hat{\boldsymbol{\theta}}_{a\,b}$$

$$\therefore \qquad \frac{\partial L_s}{\partial \dot{\boldsymbol{r}}_c} = \frac{1}{2}\,\hbar\,\alpha \sum_{a \neq b}^{N} (-\delta_{c\,a} + \delta_{c\,b})\,\frac{1}{r_{a\,b}}\,\hat{\boldsymbol{\theta}}_{a\,b}$$

$$= \frac{1}{2}\,\hbar\,\alpha \left(-\sum_{b\,(\neq c)}^{N} \frac{1}{r_{c\,b}}\,\hat{\boldsymbol{\theta}}_{c\,b} + \sum_{a\,(\neq c)}^{N} \frac{1}{r_{a\,c}}\,\hat{\boldsymbol{\theta}}_{a\,c} \right)$$

$$= \hbar\,\alpha \sum_{a\,(\neq c)}^{N} \frac{1}{r_{a\,c}}\,\hat{\boldsymbol{\theta}}_{a\,c}$$

$$= \hbar\,\alpha \sum_{a\,(\neq c)}^{N} \nabla_c\,\theta_{a\,c}$$

$$\rightarrow \qquad \boldsymbol{p}_{a\,i} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_a} = M\,\dot{\boldsymbol{r}}_a + \hbar\,\alpha \sum_{b\,(\neq a)} \nabla_a\,\theta_{b\,a} = M\,\dot{\boldsymbol{r}}_a + \hbar\,\alpha \sum_{b\,(\neq a)} \frac{1}{r_{b\,a}}\,\hat{\boldsymbol{\theta}}_{b\,a}$$

Let $\quad \boldsymbol{C}(\boldsymbol{r}_a) = \dfrac{1}{\check{q}}\,\dfrac{\partial L_S}{\partial \dot{\boldsymbol{r}}_a} = \dfrac{\hbar\,\alpha}{\check{q}} \sum_{b\,(\neq a)} \nabla_a\,\theta_{b\,a} = -\boldsymbol{C}(\boldsymbol{r}_a)_{\text{Ezawa}}$

where the “charge” $\check{q}$ was introduced for later convenience.

$$\rightarrow \qquad \boldsymbol{p}_a = M\,\dot{\boldsymbol{r}}_a + \check{q}\,\boldsymbol{C}(\boldsymbol{r}_a)$$

$$\boldsymbol{p}_a \cdot \dot{\boldsymbol{r}}_a = M\,\dot{\boldsymbol{r}}_a^2 + \hbar\,\alpha \sum_{b\,(\neq a)} \dot{\boldsymbol{r}}_a \cdot \nabla_a\,\theta_{b\,a}$$

$$= M\,\dot{\boldsymbol{r}}_a^2 + \hbar\,\alpha \sum_{b\,(\neq a)} \frac{d\,\theta_{b\,a}}{d\,t}$$

$$\therefore \qquad \sum_a \boldsymbol{p}_a \cdot \dot{\boldsymbol{r}}_a = M \sum_a \dot{\boldsymbol{r}}_a^2 + \hbar\,\alpha \sum_{b \neq a} \frac{d\,\theta_{b\,a}}{d\,t}$$

$$\rightarrow \qquad H = \sum_a \boldsymbol{p}_a \cdot \dot{\boldsymbol{r}}_a - L = \frac{1}{2}\,M \sum_a \dot{\boldsymbol{r}}_a^2 + V(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)$$

$$= \frac{1}{2\,M} \sum_a \left[\boldsymbol{p}_a - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_a)\right]^2 + V$$

$$= \frac{1}{2\,M} \sum_a \left[\frac{\hbar}{i}\,\nabla_a - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_a)\right]^2 + V$$

Comparing with the minimal coupling Hamiltonian

$$H = \frac{1}{2\,M} \sum_a \left(\boldsymbol{p}_a - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r}_a) \right)^2$$

we see that the anyon is equivalent to a ordinary particle with a “charge” $\check{q}\,c$ coupled to a gauge potential $\boldsymbol{C}$, also known as the Chern-Simons field.

For $\boldsymbol{r}$ in the xy plane,

$$\nabla \times \left(\frac{1}{r}\,\hat{\boldsymbol{\theta}} \right) = \left(\nabla \frac{1}{r} \right) \times \hat{\boldsymbol{\theta}} + \frac{1}{r}\,\nabla \times \hat{\boldsymbol{\theta}}$$

$$= -\frac{1}{r^2}\,\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} + \frac{1}{r^2}\,\hat{\boldsymbol{z}}$$

$$= 0 \qquad \forall\ r \neq 0$$

For $r = 0$, consider

$$I = \int_S d^2 r\, \nabla \times \left(\frac{1}{r} \hat{\boldsymbol{\theta}} \right) = \oint_C d\,\boldsymbol{r} \cdot \frac{1}{r} \hat{\boldsymbol{\theta}}$$

Let C be a circle of radius R around the origin.

→ $$d\,\boldsymbol{r} = \hat{\boldsymbol{\theta}}\, R\, d\,\theta$$

$$I = \int_0^{2\pi} d\,\theta = 2\pi$$

∴ $$\nabla \times \left(\frac{1}{r} \hat{\boldsymbol{\theta}} \right) = 2\pi\, \hat{\boldsymbol{z}}\, \delta^2(\boldsymbol{r})$$

We can generalize

$$\boldsymbol{C}(\boldsymbol{r}_a) = \frac{\hbar\,\alpha}{\check{q}} \sum_{b\,(\neq a)} \nabla_a \theta_{b\,a} = \frac{\hbar\,\alpha}{\check{q}} \sum_{b\,(\neq a)} \frac{1}{r_{b\,a}} \hat{\boldsymbol{\theta}}_{b\,a}$$

by replacing ∇_a with ∇ & then removing the condition $b \neq a$. Thus,

$$\boldsymbol{C}(\boldsymbol{r}) = \frac{\hbar\,\alpha}{\check{q}} \sum_a \nabla \theta_a = \frac{\hbar\,\alpha}{\check{q}} \sum_a \frac{1}{r_a(\boldsymbol{r})} \hat{\boldsymbol{\theta}}_a(\boldsymbol{r})$$

where $\theta_a(\boldsymbol{r})$ is the azimuthal angle of the vector $\boldsymbol{r}_a(\boldsymbol{r}) = \boldsymbol{r} - \boldsymbol{r}_a$.
$\hat{\boldsymbol{\theta}}_a(\boldsymbol{r})$ is the corresponding unit vector & $r_a(\boldsymbol{r}) = | \boldsymbol{r} - \boldsymbol{r}_a |$.

→ $$\nabla \times \boldsymbol{C}(\boldsymbol{r}) = \frac{\hbar\,\alpha}{\check{q}} \sum_a \nabla \times \left[\frac{1}{r_a(\boldsymbol{r})} \hat{\boldsymbol{\theta}}_a(\boldsymbol{r}) \right]$$

$$= 2\pi \frac{\hbar\,\alpha}{\check{q}} \hat{\boldsymbol{z}} \sum_a \delta^2(\boldsymbol{r} - \boldsymbol{r}_a)$$

$$= 2\pi \frac{\hbar\,\alpha}{\check{q}} \rho(\boldsymbol{r})\, \hat{\boldsymbol{z}}$$

where

$$\rho(\boldsymbol{r}) = \sum_b \delta^2(\boldsymbol{r} - \boldsymbol{r}_b)$$

is the number density of anyons.

In terms of components, one has

$$\varepsilon_{i\,j\,3}\, \partial_i\, \boldsymbol{C}_j(\boldsymbol{r}) = 2\pi \frac{\hbar\,\alpha}{\check{q}} \rho(\boldsymbol{r})$$