## 8.3.a. System of 2 Anyons

For $N = 2$,

$$H = \frac{1}{2M}\left\{\left[\boldsymbol{p}_1 - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_1)\right]^2 + \left[\boldsymbol{p}_2 - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_2)\right]^2\right\} + V(r)$$

where

$$\boldsymbol{r} = \boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \qquad\qquad r = |\,\boldsymbol{r}\,|$$

$$\boldsymbol{C}(\boldsymbol{r}_a) = \frac{\hbar\alpha}{\check{q}} \sum_{b\,(\neq a)} \frac{1}{r_{ba}} \hat{\boldsymbol{\theta}}_{ba} = \frac{\hbar\alpha}{\check{q}} \sum_{b\,(\neq a)} \nabla_a \theta_{ba}$$

$\rightarrow$
$$\boldsymbol{C}(\boldsymbol{r}_1) = \frac{\hbar\alpha}{\check{q}}\frac{1}{r}\hat{\boldsymbol{\theta}}_{21} = \frac{\hbar\alpha}{\check{q}}\nabla_1\theta_{21} = -\frac{\hbar\alpha}{\check{q}}\nabla\theta(\boldsymbol{r})$$

$$\boldsymbol{C}(\boldsymbol{r}_2) = \frac{\hbar\alpha}{\check{q}}\frac{1}{r}\hat{\boldsymbol{\theta}}_{12} = \frac{\hbar\alpha}{\check{q}}\nabla_2\theta_{12} = \frac{\hbar\alpha}{\check{q}}\nabla\theta(\boldsymbol{r})$$

Let
$$\boldsymbol{C}(\boldsymbol{r}) = \frac{\hbar\alpha}{\check{q}}\nabla\theta(\boldsymbol{r}) = \frac{\hbar\alpha}{\check{q}\,r}\hat{\boldsymbol{\theta}}$$

$\rightarrow$
$$\boldsymbol{C}(\boldsymbol{r}_2) = \boldsymbol{C}(\boldsymbol{r}) = -\boldsymbol{C}(\boldsymbol{r}_1)$$

Caution: In the formula derived in 8.3._QuantumMechanics.pdf,

$$\boldsymbol{C}(\boldsymbol{r}) = \frac{\hbar\alpha}{\check{q}}\sum_a \frac{1}{r_a(\boldsymbol{r})}\hat{\boldsymbol{\theta}}_a(\boldsymbol{r})$$

$\boldsymbol{r}$ = position vector for a point in the common coordinate system for all particles.
In the present section, $\boldsymbol{r} = \boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ denotes the relative coordinates between 2 particles.

Let

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_2 + \boldsymbol{r}_1) \qquad\qquad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$$

$\rightarrow$
$$\boldsymbol{r}_1 = \boldsymbol{R} - \frac{1}{2}\boldsymbol{r} \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} + \frac{1}{2}\boldsymbol{r}$$

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{R}} - \frac{1}{2}\dot{\boldsymbol{r}} \qquad\qquad \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{R}} + \frac{1}{2}\dot{\boldsymbol{r}}$$

$$\frac{\boldsymbol{p}_1}{M} \equiv \frac{\boldsymbol{p}_R}{m_R} - \frac{\boldsymbol{p}_r}{2\,m_r} \qquad\qquad \frac{\boldsymbol{p}_2}{M} \equiv \frac{\boldsymbol{p}_R}{m_R} + \frac{\boldsymbol{p}_r}{2\,m_r}$$

$\therefore$
$$\dot{\boldsymbol{r}}_1 = \frac{1}{M}\left[\boldsymbol{p}_1 - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_1)\right] = \frac{\boldsymbol{p}_R}{m_R} - \frac{\boldsymbol{p}_r}{2\,m_r} + \frac{1}{M}\check{q}\,\boldsymbol{C}(\boldsymbol{r})$$

$$\dot{\boldsymbol{r}}_2 = \frac{1}{M}\left[\boldsymbol{p}_2 - \check{q}\,\boldsymbol{C}(\boldsymbol{r}_2)\right] = \frac{\boldsymbol{p}_R}{m_R} + \frac{\boldsymbol{p}_r}{2\,m_r} - \frac{1}{M}\check{q}\,\boldsymbol{C}(\boldsymbol{r})$$

$\rightarrow$
$$\dot{\boldsymbol{r}}_1^2 + \dot{\boldsymbol{r}}_2^2 = \left(\frac{\boldsymbol{p}_R}{m_R}\right)^2 + \left[\frac{\boldsymbol{p}_r}{2\,m_r} - \frac{1}{M}\check{q}\,\boldsymbol{C}(\boldsymbol{r})\right]^2$$

On the other hand,

$$\frac{1}{2}M(\dot{\boldsymbol{r}}_1^2 + \dot{\boldsymbol{r}}_2^2) = M\dot{\boldsymbol{R}}^2 + \frac{1}{4}M\dot{\boldsymbol{r}}^2$$
$$\equiv \frac{1}{2}m_R\dot{\boldsymbol{R}}^2 + \frac{1}{2}m_r\dot{\boldsymbol{r}}^2$$

$\rightarrow$
$$m_R = 2M \qquad\qquad m_r = \frac{1}{2}M$$

$\therefore \quad \frac{1}{2} M(\dot{\boldsymbol{r}}_1^2 + \dot{\boldsymbol{r}}_2^2) = \frac{\boldsymbol{p}_R^2}{2 m_R} + \frac{1}{2 m_r}[\boldsymbol{p}_r - \check{q}\, \boldsymbol{C}(\boldsymbol{r})]^2$

$\rightarrow \quad H = H_R + H_r$

with $\quad H_R = \frac{\boldsymbol{p}_R^2}{2 m_R} \qquad H_r = \frac{1}{2 m_r}[\boldsymbol{p}_r - \check{q}\, \boldsymbol{C}(\boldsymbol{r})]^2 + V(r)$

$$[\boldsymbol{p}_r - \check{q}\, \boldsymbol{C}(\boldsymbol{r})]^2 \psi = \left( \frac{\hbar}{i} \nabla - \frac{\hbar \alpha}{r} \hat{\boldsymbol{\theta}} \right)^2 \psi$$

$$= -\hbar^2 \left\{ \nabla^2 \psi - i \alpha \left[ \nabla \cdot \left( \frac{\hat{\boldsymbol{\theta}}}{r} \psi \right) + \frac{\hat{\boldsymbol{\theta}}}{r} \cdot \nabla \psi \right] - \left( \frac{\alpha}{r} \right)^2 \psi \right\}$$

$$\frac{\hat{\boldsymbol{\theta}}}{r} \cdot \nabla \psi = \frac{1}{r^2} \frac{\partial \psi}{\partial \theta}$$

$$\nabla \cdot \left( \frac{\hat{\boldsymbol{\theta}}}{r} \psi \right) = \hat{\boldsymbol{\theta}} \cdot \nabla \frac{\psi}{r} + \frac{\psi}{r} \nabla \cdot \hat{\boldsymbol{\theta}} = \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \qquad ( \nabla \cdot \hat{\boldsymbol{\theta}} = 0 )$$

$$\rightarrow \quad [\boldsymbol{p}_r - \check{q}\, \boldsymbol{C}(\boldsymbol{r})]^2 \psi = -\hbar^2 \left[ \nabla^2 \psi - 2 i \alpha \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} - \left( \frac{\alpha}{r} \right)^2 \psi \right]$$

In polar coordinates, we have

$$[\boldsymbol{p}_r - \check{q}\, \boldsymbol{C}(\boldsymbol{r})]^2 \psi = -\hbar^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - 2 i \alpha \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} - \left( \frac{\alpha}{r} \right)^2 \psi \right]$$

$$= -\hbar^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} - i \alpha \right)^2 \psi \right]$$

$$\therefore \quad H_r = -\frac{\hbar^2}{2 m_r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} - i \alpha \right)^2 \right] + V(r)$$

The Schrodinger eq

$$H_r \psi = E \psi$$

is therefore separable.

Treating $\psi$ as single-valued [$\psi(r, \theta + 2\pi) = \psi(r, \theta)$], we set

$$\psi(r, \theta) = e^{i l \theta} \psi_l(r) \qquad l = 0, \pm 1, \pm 2, \ldots$$

Since $\theta \rightarrow \theta + \pi$ denotes a particle exchange, we have

$$l = \begin{cases} 0, \pm 2, \pm 4, \ldots & \text{for bosons} \\ \pm 1, \pm 3, \ldots & \text{for fermions} \end{cases}$$

The radial eq. becomes

$$-\frac{\hbar^2}{2 m_r} \frac{1}{r} \frac{d}{d r} \left( r \frac{d \psi_l}{d r} \right) + \frac{\hbar^2}{2 m_r r^2} (l - \alpha)^2 \psi_l + V \psi_l = E_l \psi_l$$

so that the eigen-energy $E_l$ is $\alpha$ dependent.

Alternatively, if we allow $\psi$ to be multi-valued & set

$$\psi(r, \theta) = \begin{cases} e^{i (l + \alpha) \theta} \psi_l(r) & \text{for bosons} \\ e^{i (l - 1 + \alpha) \theta} \psi_l(r) & \text{for fermions} \end{cases} \qquad (0 < \alpha < 1)$$

then

$$\psi(r, \theta + \pi) = \begin{cases} e^{i \alpha \pi} \psi(r, \theta) & \text{for bosons} \\ e^{-i (1 - \alpha) \pi} \psi(r, \theta) & \text{for fermions} \end{cases}$$

so that we have an anyon. Furthermore, the radial eq simplifies to

$$-\frac{\hbar^2}{2m_r}\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_l}{dr}\right)+\frac{\hbar^2}{2m_r r^2}l^2\psi_l+V\psi_l=E_l\psi_l$$

which is the same as a system without the Chern-Simons field.

Thus, if $\boldsymbol{C}$ is treated as a real physical field, the particles can be bosons or fermions & their energies depend on the strength $\alpha$ of $\boldsymbol{C}$.

If $\boldsymbol{C}$ is treated as an auxiliary field, bosons (or fermions) become anyons of statistical parameter $\alpha$ (or $1-\alpha$), but their energies remain the same as if $\alpha=0$.

## Central Harmonic Oscillator

Let $\quad V(r)=\frac{1}{2}m_r\omega^2 r^2=\frac{1}{4}M\omega^2 r^2$

$$\rightarrow\quad -\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_l}{dr}\right)+\frac{1}{r^2}(l-\alpha)^2\psi_l+\left(\frac{m_r\omega}{\hbar}\right)^2 r^2\psi_l=\frac{2m_r E_l}{\hbar^2}\psi_l$$

$$\psi_l''+\frac{1}{r}\psi_l'+\left[\epsilon_l-\frac{1}{r^2}(l-\alpha)^2-\Omega^2 r^2\right]\psi_l=0$$

where $\quad \epsilon_l=\frac{2m_r E_l}{\hbar^2} \qquad\qquad \Omega=\frac{m_r\omega}{\hbar}$

Let $\quad \psi_l=r^{|l-\alpha|}e^{-\frac{1}{2}\Omega r^2}u_l \qquad$ ( $|l-\alpha|$ is used to keep $\psi_l$ regular at $r=0$ )

$$\rightarrow\quad \psi_l'=r^{|l-\alpha|}e^{-\frac{1}{2}\Omega r^2}\left[\left(\frac{|l-\alpha|}{r}-\Omega r\right)u_l+u_l'\right]$$

$$\psi_l''=r^{|l-\alpha|}e^{-\frac{1}{2}\Omega r^2}\left\{\left(\frac{|l-\alpha|}{r}-\Omega r\right)\left[\left(\frac{|l-\alpha|}{r}-\Omega r\right)u_l+u_l'\right]\right.$$
$$\left.+\left(-\frac{|l-\alpha|}{r^2}-\Omega\right)u_l+\left(\frac{|l-\alpha|}{r}-\Omega r\right)u_l'+u_l''\right\}$$
$$=r^{|l-\alpha|}e^{-\frac{1}{2}\Omega r^2}\left\{u_l''+2\left(\frac{|l-\alpha|}{r}-\Omega r\right)u_l'+\left[\left(\frac{|l-\alpha|}{r}-\Omega r\right)^2-\left(\frac{|l-\alpha|}{r^2}+\Omega\right)\right]u_l\right\}$$

$$\therefore\quad u_l''+2\left(\frac{|l-\alpha|+\frac{1}{2}}{r}-\Omega r\right)u_l'+C\,u_l=0$$

where

$$C=\left(\frac{|l-\alpha|}{r}-\Omega r\right)^2-2\Omega+\epsilon_l-\frac{1}{r^2}(l-\alpha)^2-\Omega^2 r^2$$
$$=-2(|l-\alpha|+1)\Omega+\epsilon_l$$

$$\rightarrow\quad u_l''+2\left(\frac{|l-\alpha|+\frac{1}{2}}{r}-\Omega r\right)u_l'+[\epsilon_l-2(|l-\alpha|+1)\Omega]u_l=0$$

Let $\quad u_l=\sum_{k=0}^{\infty}a_k r^k$

$$\rightarrow\quad u_l'=\sum_{k=1}^{\infty}k\,a_k r^{k-1}=\sum_{k=0}^{\infty}(k+1)\,a_{k+1}r^k$$

$$u_l'' = \sum_{k=1}^{\infty} k\,(k+1)\,a_{k+1}\,r^{k-1} = \sum_{k=0}^{\infty} (k+1)\,(k+2)\,a_{k+2}\,r^k$$

Putting these into the diff. eq., we get the coefficient of $r^k$ as follows,

$k = -1$: $2\left(\,|\,l-\alpha\,| + \frac{1}{2}\right) a_1$

$k \geq 0$: $$(k+1)(k+2)\,a_{k+2} + 2\left(\,|\,l-\alpha\,| + \frac{1}{2}\right)(k+2)\,a_{k+2} - 2\,\Omega\,k\,a_k + [\epsilon_l - 2\,(\,|\,l-\alpha\,| + 1)\,\Omega]\,a_k$$
$$= [\,2\,|\,l-\alpha\,| + k + 2]\,(k+2)\,a_{k+2} + [\epsilon_l - 2\,(\,|\,l-\alpha\,| + k + 1)\,\Omega]\,a_k$$

Setting each to zero, we get

$$a_1 = 0$$

& $$a_{k+2} = -\frac{\epsilon_l - 2\,(\,|\,l-\alpha\,| + k + 1\,)\,\Omega}{[\,2\,|\,l-\alpha\,| + k + 2\,]\,(k+2)}\,a_k$$

$\therefore$ $$a_{2n+2} = -\frac{\epsilon_l - 2\,(\,|\,l-\alpha\,| + 2\,n + 1\,)\,\Omega}{2\,(\,|\,l-\alpha\,| + n + 1\,]\quad 2\,(n+1)}\,a_{2n} \qquad (k = 2\,n)$$

Series must terminate at some $n = n_r$.

$\rightarrow$ $\epsilon_{n_r\,l} = 2\,(\,|\,l-\alpha\,| + 2\,n_r + 1)\,\Omega$ $\qquad n_r = 0,\ 1,\ 2,\ 3,\ \ldots$

$\therefore$ $E_{n_r\,l} = \hbar\,\omega\,(\,|\,l-\alpha\,| + 2\,n_r + 1)$

Since $0 \leq \alpha < 1$ & $l = 0,\ \pm 2,\ \pm 4,\ \ldots,$ we have

$$E_{n_r\,l} = \hbar\omega \begin{cases} l - \alpha + 2\,n_r + 1 & \text{for } l > 0 \\ \alpha + 2\,n_r + 1 & \text{for } l = 0 \\ |\,l\,| + \alpha + 2\,n_r + 1 & \text{for } l < 0 \end{cases}$$

Let $n = n_r + \frac{1}{2}\left|\,l\,\right| = 0,\ 1,\ 2,\ \ldots$

$\rightarrow$ $$E_n(\alpha) = \hbar\omega \begin{cases} 2\,n + 1 - \alpha & \text{for } l > 0 \\ 2\,n + 1 + \alpha & \text{for } l \leq 0 \end{cases}$$

Note that $n,\ n_r,$ & $\frac{1}{2}\left|\,l\,\right|$ are all positive indefinite integers.

Thus, for a given $n$,

$n_r = 0,\ 1,\ \ldots,\ n-1$ if $l \geq 2$

$n_r = 0,\ 1,\ \ldots,\ n$ if $l \leq 0$

$\therefore$ Degeneracy of $E_n(\alpha) = \begin{cases} n & \text{for } l > 0 \\ n+1 & \text{for } l \leq 0 \end{cases}$

Putting everything together, the energy eigenfunctions takes the form

$$\Psi_B(\boldsymbol{R},\ r,\ \theta) \propto e^{i\,\boldsymbol{K}\cdot\boldsymbol{R}}\,e^{i\,l\,\theta}\,r^{|l-\alpha|}\,e^{-\frac{1}{2}\Omega\,r^2}\,u_l(r)$$

where $\hbar\,\boldsymbol{K}$ is the momentum of the center of mass. The energy of the state is

$$E = \frac{\hbar^2\,\boldsymbol{K}^2}{2\,m_R} + E_n(\alpha)$$

The anyonic solution is

$$\Psi_A(\boldsymbol{R},\ r,\ \theta) \propto e^{i\,\boldsymbol{K}\cdot\boldsymbol{R}}\,e^{i\,(l+\alpha)\,\theta}\,r^{|l|}\,e^{-\frac{1}{2}\Omega\,r^2}\,u_l(r)$$

with energy

$$E = \frac{\hbar^2 \boldsymbol{K}^2}{2\, m_R} + E_n(0)$$