8.4. Chern-Simons Gauge Theory

Ref: A.Khare, "Fractional Statistics & Quantum Theory", 1997, §6.2-3 & 8.2.
(Khare uses the same metric as Ezawa.)

Chern-Simons Term

In 2 + 1 dimensions, the Chern-Simons term is given by

$$\mathcal{L}_{CS} = -\frac{1}{4} m_C \varepsilon^{\mu\nu\lambda 3} F_{\mu\nu} C_\lambda \qquad F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$$

$$= -\frac{1}{2} m_C \varepsilon^{\lambda\mu\nu 3} C_\lambda \partial_\mu C_\nu$$

where m_C is the "mass" (in natural units $\hbar = c = 1$) of the gauge field C_μ.

$$\varepsilon^{\lambda\mu\nu 3} C_\lambda \partial_\mu C_\nu = \varepsilon^{0\mu\nu 3} C_0 \partial_\mu C_\nu + \varepsilon^{i\mu\nu 3} C_i \partial_\mu C_\nu$$

$$= \varepsilon^{0ij3} C_0 \partial_i C_j + \varepsilon^{i0\nu 3} C_i \partial_0 C_\nu + \varepsilon^{ij\nu 3} C_i \partial_j C_\nu$$

$$= \varepsilon^{0ij3} C_0 \partial_i C_j + \varepsilon^{i0j3} C_i \partial_0 C_j + \varepsilon^{ij03} C_i \partial_j C_0$$

$$= \varepsilon^{0ij3} \left[C_0 \partial_i C_j + C_i \left(-\partial_0 C_j + \partial_j C_0 \right) \right]$$

$$= -\varepsilon_{ij3} \left[C_0 \partial_i \boldsymbol{C}_j + \boldsymbol{C}_i \left(\partial_0 \boldsymbol{C}_j + \partial_j C_0 \right) \right]$$

$$= -[C_0 \nabla \times \boldsymbol{C} + \boldsymbol{C} \times (\partial_0 \boldsymbol{C} + \nabla C_0)]_3$$

∴

$$\mathcal{L}_{CS} = \frac{1}{2} m_C \varepsilon_{ij3} \left[C_0 \partial_i \boldsymbol{C}_j + \boldsymbol{C}_i \left(\partial_0 \boldsymbol{C}_j + \partial_j C_0 \right) \right]$$

$$= \frac{1}{2} m_C [C_0 \nabla \times \boldsymbol{C} + \boldsymbol{C} \times (\partial_0 \boldsymbol{C} + \nabla C_0)]_3$$

Anyon $\mathcal{H}$ & $\mathcal{L}$

The anyon Hamiltonian density in 2-D space is

$$\mathcal{H} = \frac{1}{2M} \left(\frac{\hbar}{i} \nabla + \check{q} \boldsymbol{C} \right) \phi^+ \left(\frac{\hbar}{i} \nabla - \check{q} \boldsymbol{C} \right) \phi + \mathcal{V}$$

$$= -\frac{\hbar^2}{2M} \left(\nabla + i \frac{\check{q}}{\hbar} \boldsymbol{C} \right) \phi^+ \left(\nabla - i \frac{\check{q}}{\hbar} \boldsymbol{C} \right) \phi + \mathcal{V}$$

where

$$\mathcal{V}(x) = V(\boldsymbol{r}) \phi^+(x) \phi(x) + \frac{1}{2} \int d^2 r' \phi^+(x) \phi^+(x') U(\boldsymbol{r} - \boldsymbol{r}') \phi(x') \phi(x)$$

& the Chern-Simons field $\boldsymbol{C}(x)$ is subject to the constraint

$$\varepsilon_{ij3} \partial_i \boldsymbol{C}_j = \frac{2\pi\hbar\alpha}{\check{q}} \phi^+ \phi$$

Let $\quad D_\mu = \partial_\mu + i \frac{\check{q}}{\hbar} C_\mu = \left(\frac{1}{c} \partial_t + i \frac{\check{q}}{\hbar} C_0, \; \nabla - i \frac{\check{q}}{\hbar} \boldsymbol{C} \right) = (D_0, -\boldsymbol{D})$

→ $\quad \mathcal{H} = \frac{\hbar^2}{2M} \boldsymbol{D}^* \phi^+ \cdot \boldsymbol{D} \phi + \mathcal{V}$

$$\nabla \cdot \left(\phi^+ \boldsymbol{D} \phi \right) = \nabla \phi^+ \cdot \boldsymbol{D} \phi + \phi^+ \nabla \cdot (\boldsymbol{D} \phi) = \nabla \phi^+ \cdot \boldsymbol{D} \phi + i \frac{\check{q}}{\hbar} \boldsymbol{C} \phi^+ \cdot \boldsymbol{D} \phi - \phi^+ i \frac{\check{q}}{\hbar} \boldsymbol{C} \cdot \boldsymbol{D} \phi + \phi^+ \nabla \cdot (\boldsymbol{D} \phi)$$

$$= \boldsymbol{D}^* \phi^+ \cdot \boldsymbol{D} \phi + \phi^+ \boldsymbol{D} \cdot (\boldsymbol{D} \phi)$$

$$\rightarrow \qquad \mathcal{H} = -\frac{\hbar^2}{2M}\phi^+ \boldsymbol{D}^2 \phi + \mathcal{V} + \frac{\hbar^2}{2M}\nabla\cdot\left(\phi^+ \boldsymbol{D}\,\phi\right)$$

where the divergence term has no dynamical significance & can be dropped.

The Lagrangian density that gives rise to $\mathcal{H}$ together with its constraint is (c.f. 3.2._SchrodingerField-.pdf)

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{CS}$$

where

$$\mathcal{L}_0 = i\,\hbar\,c\,\phi^+ D_0\,\phi + \frac{\hbar^2}{2M}\phi^+ \boldsymbol{D}^2 \phi - \mathcal{V}$$

gives the unconstrainted $\mathcal{H}$ if $C_0 = 0$.

In the following, we'll show that $\mathcal{L}_{CS}$ turns the constraint into the Euler eq. for C_0.

Owing to the nonrelativistic nature of $\mathcal{L}$, it is easier to work with 3-D vectors $\boldsymbol{A}$ instead of A_j. Hence,

$$\mathcal{L} = i\,\hbar\,\phi^+\left(\partial_t + i\,\frac{\check{q}\,c}{\hbar}C_0\right)\phi + \frac{\hbar^2}{2M}\phi^+\left(\nabla - i\,\frac{\check{q}}{\hbar}\boldsymbol{C}\right)^2\phi - \mathcal{V} + \frac{1}{2}m_C\,\varepsilon_{ij3}\left[C_0\,\partial_i \boldsymbol{C}_j + \boldsymbol{C}_i\left(\partial_0 \boldsymbol{C}_j + \partial_j C_0\right)\right]$$

$$\frac{\partial\mathcal{L}}{\partial C_0} = -\check{q}\,c\,\phi^+\phi + \frac{1}{2}m_C\,\varepsilon_{ij3}\,\partial_i \boldsymbol{C}_j$$

$$\frac{\partial\mathcal{L}}{\partial\,\partial_0 C_0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\,\partial_j C_0} = \frac{1}{2}m_C\,\varepsilon_{ij3}\,\boldsymbol{C}_i = -\frac{1}{2}m_C\,\varepsilon_{ji3}\,\boldsymbol{C}_i$$

Euler eq. for C_0 is

$$-\check{q}\,c\,\phi^+\phi + m_C\,\varepsilon_{ij3}\,\partial_i \boldsymbol{C}_j = 0$$

Comparing with the constraint

$$\varepsilon_{ij3}\,\partial_i \boldsymbol{C}_j = \frac{2\pi\hbar\alpha}{\check{q}}\phi^+\phi$$

we have

$$m_C = \frac{\check{q}^2 c}{2\pi\hbar\alpha} \qquad\qquad \varepsilon_{ij3}\,\partial_i \boldsymbol{C}_j = \frac{\check{q}\,c}{m_C}\phi^+\phi$$

& $$\mathcal{L}_{CS} = -\frac{\check{q}^2 c}{4\pi\hbar\alpha}\varepsilon^{\lambda\mu\nu 3}C_\lambda\,\partial_\mu C_\nu = \frac{\check{q}^2 c}{4\pi\hbar\alpha}\varepsilon_{ij3}\left[C_0\,\partial_i \boldsymbol{C}_j + \boldsymbol{C}_i\left(\partial_0 \boldsymbol{C}_j + \partial_j C_0\right)\right]$$

$$\frac{\partial\mathcal{L}}{\partial \boldsymbol{C}_k} = -i\,\frac{\check{q}\,\hbar}{M}\phi^+\left(\partial_k - i\,\frac{\check{q}}{\hbar}\boldsymbol{C}_k\right)\phi + \frac{1}{2}m_C\,\varepsilon_{kj3}\left(\partial_0 \boldsymbol{C}_j + \partial_j C_0\right)$$

$$\frac{\partial\mathcal{L}}{\partial\,\partial_0 \boldsymbol{C}_k} = \frac{1}{2}m_C\,\varepsilon_{ik3}\,\boldsymbol{C}_i$$

$$\frac{\partial\mathcal{L}}{\partial\,\partial_j \boldsymbol{C}_k} = \frac{1}{2}m_C\,C_0\,\varepsilon_{jk3}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\,\partial_\mu \boldsymbol{C}_k} = -\frac{1}{2}m_C\,\varepsilon_{kj3}\left(\partial_0 \boldsymbol{C}_j + \partial_j C_0\right)$$

Euler eq. fot $\boldsymbol{C}$ is

$$-i\,\frac{\check{q}\,\hbar}{M}\phi^+\left(\partial_k - i\,\frac{\check{q}}{\hbar}\boldsymbol{C}_k\right)\phi + m_C\,\varepsilon_{kj3}\left(\partial_0 \boldsymbol{C}_j + \partial_j C_0\right) = 0$$

$$\varepsilon_{km3}\,\varepsilon_{kj3} = \delta_{mj}$$

$$\rightarrow \qquad \partial_j C_0 = -\partial_0 \boldsymbol{C}_j + i \frac{\check{q}\,\hbar}{M m_C} \varepsilon_{k\,j\,3}\, \phi^+ \left(\partial_k - i \frac{\check{q}}{\hbar} \boldsymbol{C}_k\right) \phi$$

$$\nabla C_0 = -\partial_0 \boldsymbol{C} + i \frac{2\pi \hbar^2 \alpha}{M \check{q}\, c} \phi^+ \hat{\boldsymbol{z}} \times \left(\nabla - i \frac{\check{q}}{\hbar} \check{q}\, \boldsymbol{C}\right) \phi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = i\,\hbar\, \phi^+ \qquad\qquad \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{C}}_k} = \frac{1}{2c} m_C\, \varepsilon_{i\,k\,3}\, \boldsymbol{C}_i$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{C}}} \cdot \dot{\boldsymbol{C}} = \frac{1}{2c} m_C \left(\boldsymbol{C} \times \dot{\boldsymbol{C}}\right)_3$$

$$\rightarrow \qquad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{C}}} \cdot \dot{\boldsymbol{C}} - \mathcal{L}$$

$$= \check{q}\, c\, C_0\, \phi^+ \phi + \frac{1}{2c} m_C \left(\boldsymbol{C} \times \dot{\boldsymbol{C}}\right)_3 - \frac{\hbar^2}{2M} \phi^+ \boldsymbol{D}^2 \phi + \mathcal{V} - \frac{1}{2} m_C [C_0 \nabla \times \boldsymbol{C} + \boldsymbol{C} \times (\partial_0 \boldsymbol{C} + \nabla C_0)]_3$$

$$= -\frac{\hbar^2}{2M} \phi^+ \boldsymbol{D}^2 \phi + \mathcal{V} - \frac{1}{2} m_C\, (C_0 \nabla \times \boldsymbol{C} + \boldsymbol{C} \times \nabla C_0)_3 + \check{q}\, c\, C_0\, \phi^+ \phi$$

$$\varepsilon_{i\,j\,3}\, \partial_i \boldsymbol{C}_j = \frac{\check{q}\, c}{m_C} \phi^+ \phi$$

$$\rightarrow \qquad \check{q}\, c\, C_0\, \phi^+ \phi = m_C\, C_0\, (\nabla \times \boldsymbol{C})_3$$

$$\therefore \qquad \mathcal{H} = -\frac{\hbar^2}{2M} \phi^+ \boldsymbol{D}^2 \phi + \mathcal{V} + \frac{1}{2} m_C\, (C_0 \nabla \times \boldsymbol{C} - \boldsymbol{C} \times \nabla C_0)_3$$

$$= -\frac{\hbar^2}{2M} \phi^+ \boldsymbol{D}^2 \phi + \mathcal{V} + \frac{1}{2} m_C\, [\nabla \times (C_0\, \boldsymbol{C})]_3$$

The last term can be dropped since it does not contribute to $H = \int d^2 r\, \mathcal{H}$.

$\therefore$ $\mathcal{L}$ does give rise to $\mathcal{H}$.

Canonical quantization of the field is achieved by setting

$$[\phi(t, \boldsymbol{r}), \pi(t, \boldsymbol{r}')]_{\mp} = i\,\hbar\, \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad \text{for} \quad \begin{matrix}\text{bosons} \\ \text{fermions}\end{matrix}$$

i.e., $\quad [\phi(t, \boldsymbol{r}), \phi^+(t, \boldsymbol{r}')]_{\mp} = \delta(\boldsymbol{r} - \boldsymbol{r}')$

& $\quad [\phi(t, \boldsymbol{r}), \phi(t, \boldsymbol{r}')]_{\mp} = [\phi^+(t, \boldsymbol{r}), \phi^+(t, \boldsymbol{r}')]_{\mp} = 0$