8.4.a. Equivalence to Quantum Mechanics System

In this section, everything is evaluated at the same time t, e.g.,

$$x=(t,\boldsymbol{r}) \qquad x'=(t,\boldsymbol{r}') \qquad x_a=(t,\boldsymbol{r}_a)$$

The Problem

We wish to verify the equivalence between the QM anyons described by

$$H^{QM}=\frac{1}{2M}\sum_{a=1}^{N}\boldsymbol{D}_a^2+\sum_{a<b}^{N}U(\boldsymbol{r}_b-\boldsymbol{r}_a) \qquad \boldsymbol{D}=\left(\nabla-i\frac{\check{q}}{\hbar}\boldsymbol{C}\right)$$

$$\boldsymbol{C}(\boldsymbol{r}_a)=\frac{\hbar\alpha}{\check{q}}\sum_{b\,(\neq a)}\nabla_a\theta_{ba}$$

& the Chern-Simons gauge theory with

$$H^{FT}=\int d^2r\,\mathcal{H}$$

$$\mathcal{H}=-\frac{\hbar^2}{2M}\phi^+\cdot\boldsymbol{D}^2\phi+\mathcal{V}$$

$$\mathcal{V}(x)=\frac{1}{2}\int d^2r'\,\phi^+(x)\,\phi^+(x')\,U(\boldsymbol{r}-\boldsymbol{r}')\,\phi(x')\,\phi(x)$$

& constraint

$$\varepsilon_{ij3}\,\partial_i\boldsymbol{C}_j=\frac{2\pi\hbar\alpha}{\check{q}}\phi^+\phi$$

In general, if the energy eigenstate of an N-particle system is $|\,E,N\,\rangle$, i.e.,

$$H^{FT}\left|\,E,N\right\rangle=E\,|\,E,N\rangle$$

the corresponding time-independent Schrodinger eq. is

$$H_N^{QM}\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)=E\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)$$

By definition, $\phi^+(x)$ creates a particle at x. Hence,

$$|\,\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N\,\rangle=\phi^+(\boldsymbol{r}_1)\ldots\phi^+(\boldsymbol{r}_N)\,\left|\,0\right\rangle$$

$$\therefore\qquad \Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)\equiv\langle\,\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N\,|\,E,N\,\rangle$$

$$=\langle\,0\,|\,\phi(\boldsymbol{r}_1)\ldots\phi(\boldsymbol{r}_N)\,|\,E,N\,\rangle$$

Thus,

$$\left\langle\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N\,\middle|\,H^{FT}\,\middle|\,E,N\right\rangle=E\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)$$

$$\to\qquad \left\langle 0\,\middle|\,\phi(\boldsymbol{r}_1)\ldots\phi(\boldsymbol{r}_N)\,H^{FT}\,\middle|\,E,N\right\rangle=H_N^{QM}\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)$$

Although this formally relates H_N^{QM} to H^{FT}, it has little practical value.

The vacuum is defined by

$$H^{FT}\left|\,0\right\rangle=E_0\,|\,0\rangle \qquad \& \qquad \phi(\boldsymbol{r})\,|\,0\,\rangle=0$$

Thus

$$\left\langle 0\,\middle|\left[\phi(\boldsymbol{r}_1)\ldots\phi(\boldsymbol{r}_N),\,H^{FT}\right]\middle|\,E,N\right\rangle=(E-E_0)\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)$$

Setting $E_0=0$, we have

$$H_N^{QM}\,\Psi_E(\boldsymbol{r}_1,\ \ldots,\ \boldsymbol{r}_N)=\left\langle 0\,\middle|\left[\phi(\boldsymbol{r}_1)\ldots\phi(\boldsymbol{r}_N),\,H^{FT}\right]\middle|\,E,N\right\rangle$$

Our task is to prove this for the anyons.

Note that

$$\langle 0 \mid \phi^+ = 0$$

means any operator with ϕ^+ as the leftmost factor does not contribute to H_N^{QM}.

Incidentally, this means that if we define

$$E_0 = \langle 0 \mid H^{FT} \mid 0 \rangle$$

then E_0 must be 0, i.e., the origin of the energy scale in field theory is fixed.

Proof for Case **C** = 0

For the ordinary N-particle system

$$H^{QM} = \frac{1}{2M} \sum_{a=1}^{N} \nabla_a^2 + \sum_{a<b}^{N} U(\boldsymbol{r}_b - \boldsymbol{r}_a)$$

& $$\mathcal{H} = -\frac{\hbar^2}{2M} \phi^+ \cdot \nabla^2 \phi + \mathcal{V}$$

the proof is easy.

Let $$\mathbb{K} = -\frac{\hbar^2}{2M} \int d^2 r' \, \phi^+(x') \nabla'^2 \phi(x')$$

$$\mathbb{V} = \int d^2 r' \, \mathcal{V}(x')$$

→ $$H^{FT} = \mathbb{K} + \mathbb{V}$$

$$[\phi(x), \phi^+(x')] = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad [\phi(x), \phi(x')] = [\phi^+(x), \phi^+(x')] = 0$$

→ $$[\phi(x), \mathbb{K}] = -\frac{\hbar^2}{2M} \int d^2 r' \, \delta(\boldsymbol{r} - \boldsymbol{r}') \nabla'^2 \phi(x')$$
$$= -\frac{\hbar^2}{2M} \nabla^2 \phi(x)$$

∴ $$[\phi(x_1)\, \phi(x_2), \mathbb{K}] = [\phi(x_1), \mathbb{K}]\, \phi(x_2) + \phi(x_1)\, [\phi(x_2), \mathbb{K}]$$
$$= -\frac{\hbar^2}{2M} \left\{ \nabla_1^2 \phi(x_1)\, \phi(x_2) + \phi(x_1)\, \nabla_2^2 \phi(x_2) \right\}$$
$$= -\frac{\hbar^2}{2M} \left\{ \nabla_1^2 + \nabla_2^2 \right\} \phi(x_1)\, \phi(x_2)$$

$$[\phi(x), \mathbb{V}] = \frac{1}{2} \int d^2 r' \int d^2 r'' [\phi(x), \phi^+(x')\, \phi^+(x'')]\, U(\boldsymbol{r}' - \boldsymbol{r}'')\, \phi(x'')\, \phi(x')$$
$$= \frac{1}{2} \int d^2 r' \int d^2 r'' \{ \delta(\boldsymbol{r} - \boldsymbol{r}')\, \phi^+(x'') + \phi^+(x')\, \delta(\boldsymbol{r} - \boldsymbol{r}'') \}\, U(\boldsymbol{r}' - \boldsymbol{r}'')\, \phi(x'')\, \phi(x')$$
$$= \frac{1}{2} \left\{ \int d^2 r'' \, \phi^+(x'')\, U(\boldsymbol{r} - \boldsymbol{r}'')\, \phi(x'')\, \phi(x) + \int d^2 r' \, \phi^+(x')\, U(\boldsymbol{r}' - \boldsymbol{r})\, \phi(x)\, \phi(x') \right\}$$
$$= \int d^2 r' \, \phi^+(x')\, U(\boldsymbol{r} - \boldsymbol{r}')\, \phi(x')\, \phi(x)$$
$$= \overline{U}(x)\, \phi(x)$$

where

$$\overline{U}(x) = \int d^2 r' \, \phi^+(x')\, U(\boldsymbol{r} - \boldsymbol{r}')\, \phi(x')$$

$$= \int d^2 r' \, \rho(x') \, U(\boldsymbol{r} - \boldsymbol{r}')$$

$\rightarrow \quad \langle 0 \mid \overline{U} = 0$

$$\therefore \quad [\phi(x_1)\, \phi(x_2),\, \mathbb{V}] = [\phi(x_1),\, \mathbb{V}]\, \phi(x_2) + \phi(x_1)\, [\phi(x_2),\, \mathbb{V}]$$
$$= \overline{U}(x_1)\, \phi(x_1)\, \phi(x_2) + \phi(x_1)\, \overline{U}(x_2)\, \phi(x_2)$$

$$\phi(x_1)\, \overline{U}(x_2) = \phi(x_1) \int d^2 r' \, \phi^+(x')\, U(\boldsymbol{r}_2 - \boldsymbol{r}')\, \phi(x')$$
$$= \int d^2 r' \{ \delta(\boldsymbol{r}_1 - \boldsymbol{r}') + \phi^+(x')\, \phi(x_1) \}\, U(\boldsymbol{r}_2 - \boldsymbol{r}')\, \phi(x')$$
$$= U(\boldsymbol{r}_2 - \boldsymbol{r}_1)\, \phi(x_1) + \overline{U}(x_2)\, \phi(x_1)$$

$$\therefore \quad [\phi(x_1)\, \phi(x_2),\, \mathbb{V}] = \left\{ \overline{U}(x_1) + \overline{U}(x_2) + U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \right\} \phi(x_1)\, \phi(x_2)$$

Since

$$\langle 0 \mid \overline{U} = 0$$

only the $U(\boldsymbol{r}_2 - \boldsymbol{r}_1)$ term contributes to H_N^{QM}.

1-particle system:

$$\left\langle 0 \left| \left[\phi(x),\, H^{FT} \right] \right| E,\, 1 \right\rangle = \langle 0 \mid [\phi(x),\, \mathbb{K} + \mathbb{V}] \mid E,\, 1 \rangle$$

$$\langle 0 \mid [\phi(x),\, \mathbb{K}] \mid E,\, 1 \rangle = -\frac{\hbar^2}{2M} \nabla^2 \langle 0 \mid \phi(x) \mid E,\, 1 \rangle$$
$$= -\frac{\hbar^2}{2M} \nabla^2 \psi_E(x)$$

$$\langle 0 \mid [\phi(x),\, \mathbb{V}] \mid E,\, 1 \rangle = 0$$

$$\rightarrow \quad \left\langle 0 \left| \left[\phi(x),\, H^{FT} \right] \right| E,\, 1 \right\rangle = -\frac{\hbar^2}{2M} \nabla^2 (x)\, \psi_E(x)$$
$$= H_1^{QM}\, \Psi_E(x)$$

2-particle system:

$$\langle 0 \mid [\phi(x_1)\, \phi(x_2),\, \mathbb{K}] \mid E,\, 2 \rangle = -\frac{\hbar^2}{2M} \left(\nabla_1^2 + \nabla_2^2 \right) \langle 0 \mid \phi(x_1)\, \phi(x_2) \mid E,\, 2 \rangle$$
$$= -\frac{\hbar^2}{2M} \left(\nabla_1^2 + \nabla_2^2 \right) \Psi_E(x_1,\, x_2)$$

$$\langle 0 \mid [\phi(x_1)\, \phi(x_2),\, \mathbb{V}] \mid E,\, 2 \rangle = \left\langle 0 \left| \left\{ \overline{U}(x_1) + \overline{U}(x_2) + U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \right\} \phi(x_1)\, \phi(x_2) \right| E,\, 2 \right\rangle$$
$$= U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \langle 0 \mid \phi(x_1)\, \phi(x_2) \mid E,\, 2 \rangle$$
$$= U(\boldsymbol{r}_2 - \boldsymbol{r}_1)\, \Psi_E(x_1,\, x_2)$$

$$\rightarrow \quad \left\langle 0 \left| \left[\phi(x_1)\, \phi(x_2),\, H^{FT} \right] \right| E,\, 2 \right\rangle = \left[-\frac{\hbar^2}{2M} \left(\nabla_1^2 + \nabla_2^2 \right) + U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \right] \Psi_E(x_1,\, x_2)$$
$$= H_2^{QM}\, \Psi_E(x_1,\, x_2)$$

With a little labor, one can prove the general N-particle case by the method of induction.

Proof for Case **C** ≠ 0

Ref: Khare, §8.2.

To begin, we rewrite the FT constraint as

$$(\nabla \times \boldsymbol{C})_3 = 2\pi\,\overline{\alpha}\,\rho \qquad \text{where} \qquad \overline{\alpha} = \frac{\hbar\,\alpha}{\check{q}} \qquad (\rho = \phi^+ \phi)$$

Using (see 8.3._QuantumMechanics.pdf)

$$\nabla\theta(\boldsymbol{r}) = \frac{1}{r}\,\hat{\boldsymbol{\theta}}(\boldsymbol{r}) \quad \& \qquad \nabla \times \left(\frac{1}{r}\,\hat{\boldsymbol{\theta}}\right) = 2\pi\,\hat{\boldsymbol{z}}\,\delta^2(\boldsymbol{r})$$

we see that

$$\boldsymbol{C}(x) = \overline{\alpha}\int d^2 r'\,\nabla\theta(\boldsymbol{r} - \boldsymbol{r}')\,\rho(x')$$
$$= \overline{\alpha}\int d^2 r'\,\nabla\theta(\boldsymbol{r} - \boldsymbol{r}')\,\phi^+(x')\,\phi(x')$$

$\rightarrow \qquad \langle 0 \mid \boldsymbol{C} = 0$

Compare with

$$\boldsymbol{C}(\boldsymbol{r}_a) = \overline{\alpha}\sum_{b\,(\neq a)} \nabla_a\,\theta_{b\,a} = \overline{\alpha}\sum_{b\,(\neq a)} \nabla_a\,\theta(\boldsymbol{r}_a - \boldsymbol{r}_b)$$

we see that the field theory $\boldsymbol{C}(x)$ operator is consistent with QM expression

$$\boldsymbol{C}(\boldsymbol{r}) = \overline{\alpha}\sum_a \nabla\theta(\boldsymbol{r} - \boldsymbol{r}_a)$$

where the exclusion of the term $b = a$ can be handled by setting

$$\nabla\theta(0) = \lim_{r\to 0}\frac{1}{r}\,\hat{\boldsymbol{\theta}}(\boldsymbol{r}) = 0$$

(Khare obtained this by means of regularization. See Khare, eq.8.15.)

$$[\,\phi(x),\,\boldsymbol{C}(x')\,] = \overline{\alpha}\int d^2 r_a\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r}_a)\,[\phi(x),\,\phi^+(x_a)\,\phi(x_a)]$$
$$= \overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\,\phi(x)$$

$$\rightarrow \qquad \left[\phi(\boldsymbol{r}),\,\boldsymbol{C}^2(x')\right] = \overline{\alpha}^2\int d^2 r_a\int d^2 r_b\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r}_a)\cdot\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r}_b)\,[\phi(x),\,\phi^+(x_a)\,\phi(x_a)\,\phi^+(x_b)\,\phi(x_b)]$$

$$[\phi(x),\,\phi^+(x_a)\,\phi(x_a)\,\phi^+(x_b)\,\phi(x_b)] = \delta(\boldsymbol{r} - \boldsymbol{r}_a)\,\phi(x)\,\phi^+(x_b)\,\phi(x_b) + \delta(\boldsymbol{r} - \boldsymbol{r}_b)\,\phi^+(x_a)\,\phi(x_a)\,\phi(x)$$

$$\rightarrow \qquad \left[\phi(x),\,\boldsymbol{C}^2(x')\right] = \overline{\alpha}^2\left\{\phi(x)\int d^2 r_b\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\cdot\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r}_b)\,\rho(x_b)\right.$$
$$\left. + \int d^2 r_a\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r}_a)\cdot\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\,\rho(x_a)\,\phi(x)\right\}$$
$$= \overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\cdot\{\,\phi(x)\,\boldsymbol{C}(x') + \boldsymbol{C}(x')\,\phi(x)\,\}$$
$$= \overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\cdot\{\,\overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\,\phi(x) + 2\,\boldsymbol{C}(x')\,\phi(x)\,\}$$
$$= \overline{\alpha}^{\,2}\,[\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\,]^2\,\phi(x) + 2\,\overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\cdot\boldsymbol{C}(x')\,\phi(x)$$

$$\therefore \qquad \langle 0 \mid \left[\phi(x),\,\boldsymbol{C}^2(x')\right] = \overline{\alpha}^{\,2}\,[\,\nabla'\,\theta(\boldsymbol{r}' - \boldsymbol{r})\,]^2\,\langle 0 \mid \phi(x)$$

$$\boldsymbol{D}^2\,\phi = \left(\nabla - i\,\frac{\check{q}}{\hbar}\,\boldsymbol{C}\right)\cdot\left(\nabla\phi - i\,\frac{\check{q}}{\hbar}\,\boldsymbol{C}\phi\right)$$
$$= \nabla^2\phi - i\,\frac{\check{q}}{\hbar}[\,\nabla\cdot(\boldsymbol{C}\phi) + \boldsymbol{C}\cdot\nabla\phi\,] - \left(\frac{\check{q}}{\hbar}\right)^2\boldsymbol{C}^2\,\phi$$
$$= \nabla^2\phi - i\,\frac{\check{q}}{\hbar}[\,(\nabla\cdot\boldsymbol{C})\,\phi + 2\,\boldsymbol{C}\cdot\nabla\phi\,] - \left(\frac{\check{q}}{\hbar}\right)^2\boldsymbol{C}^2\,\phi$$
$$= \nabla^2\phi + \mathbb{C}\,\phi$$

where

$$\mathbb{C} = -i\,\frac{\check{q}}{\hbar}[\,\nabla\cdot\boldsymbol{C} + 2\,\boldsymbol{C}\cdot\nabla\,] - \left(\frac{\check{q}}{\hbar}\right)^2\boldsymbol{C}^2$$

→ $\langle 0 \,|\, \mathbb{C} = 0$

As in the EM case, the gauge field $\boldsymbol{C}$ is determined only up to a gauge transformation. The choice

$$\nabla \cdot \boldsymbol{C} = 0$$

is known as the Weyl gauge in this context. Thus,

$$\mathbb{C}\,\phi = -2\,i\,\frac{\check{q}}{\hbar}\,\boldsymbol{C}\cdot\nabla\phi - \left(\frac{\check{q}}{\hbar}\right)^2 \boldsymbol{C}^2\,\phi$$

$$[\phi(x),\,\mathbb{C}(x')\,\phi(x')\,] = -\left(2\,i\,\frac{\check{q}}{\hbar}\,\overline{\alpha}\,\nabla'\,\theta(\boldsymbol{r}'-\boldsymbol{r})\cdot\nabla' + \left(\frac{\check{q}}{\hbar}\right)^2 \overline{\alpha}^{\,2}\,[\nabla'\,\theta(\boldsymbol{r}'-\boldsymbol{r})\,]^2\right)\phi(x)\,\phi(x')$$

Let $\quad \mathbb{K} = -\dfrac{\hbar^2}{2\,M}\displaystyle\int d^2 r'\,\phi^+(x')\,\boldsymbol{D}'^{\,2}\,\phi(x')$

→ $\quad [\phi(x),\,\mathbb{K}\,] = -\dfrac{\hbar^2}{2\,M}\,\boldsymbol{D}^2(x)\,\phi(x) + \Delta\mathrm{K}(x)$

where

$$\Delta\mathrm{K}(x) = -\frac{\hbar^2}{2\,M}\int d^2 r'\,\phi^+(x')\left[\,\phi(x),\,\boldsymbol{D}'^{\,2}\,\phi(x')\right]$$

$$= -\frac{\hbar^2}{2\,M}\int d^2 r'\,\phi^+(x')\,[\,\phi(x),\,\mathbb{C}(x')\,\phi(x')]$$

→ $\quad \langle 0 \,|\, \Delta\mathrm{K} = 0$

$$\left\langle 0 \,\right|\, \boldsymbol{D}^2(x)\,\phi(x) = \left\langle 0 \,\right|\, \nabla^2\,\phi(x) + \langle 0 \,|\, \mathbb{C}(x)\,\phi(x)$$

$$= \nabla^2\,\langle 0 \,|\, \phi(x)$$

$$[\phi\,(x_1)\,\phi(x_2),\,\mathbb{K}\,] = [\phi(x_1),\,\mathbb{K}]\,\phi(x_2) + \phi(x_1)\,[\phi(x_2),\,\mathbb{K}\,]$$

$$= \left[-\frac{\hbar^2}{2\,M}\,\boldsymbol{D}_1^2\,\phi(x_1) + \Delta\mathrm{K}\,(x_1)\right]\phi(x_2) + \phi(x_1)\left[-\frac{\hbar^2}{2\,M}\,\boldsymbol{D}_2^2\,\phi(x_2) + \Delta\mathrm{K}(x_2)\right]$$

$$\phi(x_1)\,\boldsymbol{D}_2^2\,\phi(x_2) = \boldsymbol{D}_2^2\,\phi(x_1)\,\phi(x_2) + \left[\phi(x_1),\,\boldsymbol{D}_2^2\,\phi(x_2)\right]$$

$$= \boldsymbol{D}_2^2\,\phi(x_1)\,\phi(x_2) + [\phi(x_1),\,\mathbb{C}(x_2)]\,\phi(x_2)$$

→ $\quad \langle 0 |\, [\phi(x_1)\,\phi(x_2),\,\mathbb{K}\,] = -\dfrac{\hbar^2}{2\,M}\left(\nabla_1^2 + \nabla_2^2\right)\langle 0 \,|\, \phi(x_1)\,\phi(x_2)$

$$+ \langle 0 \,|\, \{\,\phi(x_1)\,\Delta\mathrm{K}(x_2) + [\phi(x_1),\,\mathbb{C}(x_2)]\,\phi(x_2)\,\}$$

$$\phi(x_1)\,\Delta\mathrm{K}(x_2) = -\frac{\hbar^2}{2\,M}\,\phi(x_1)\int d^2 r'\,\phi^+(x')\,[\,\phi\,(x_2),\,\mathbb{C}(x')\,\phi(x')]$$

$$= -\frac{\hbar^2}{2\,M}\left\{\int d^2 r'\,\phi^+(x')\,\phi(x_1)\,[\,\phi\,(x_2),\,\mathbb{C}(x')\,\phi(x')] + [\,\phi\,(x_2),\,\mathbb{C}\,(x_1)\,\phi\,(x_1)]\right\}$$

→ $\quad \langle 0 |\, \phi(x_1)\,\Delta\mathrm{K}(x_2) = \dfrac{\hbar^2}{2\,M}\left(2\,i\,\dfrac{\check{q}}{\hbar}\,\overline{\alpha}\,\nabla_2\,\theta\,(\boldsymbol{r}_2-\boldsymbol{r}_1)\cdot\nabla_2 + \left(\dfrac{\check{q}}{\hbar}\right)^2 \overline{\alpha}^{\,2}\,[\nabla_2\,\theta(\boldsymbol{r}_2-\boldsymbol{r}_1)\,]^2\right)\langle 0 \,|\, \phi(x_1)\,\phi\,(x_2)$

$$= \frac{\hbar^2}{2\,M}\left(2\,i\,\frac{\check{q}}{\hbar}\,\overline{\alpha}\,\nabla_2\,\theta\,(\boldsymbol{r}_2-\boldsymbol{r}_1)\cdot\nabla_2 + \left(\frac{\check{q}}{\hbar}\right)^2 \overline{\alpha}^{\,2}\,[\nabla_2\,\theta(\boldsymbol{r}_2-\boldsymbol{r}_1)\,]^2\right)\langle 0 \,|\, \phi(x_1)\,\phi\,(x_2)$$

1-Particle System

$$\left\langle 0 \,\right|\, \left[\,\phi(x),\,H^{\mathrm{FT}}\right] \left|\, E,\,1\,\right\rangle = \langle 0 \,|\, [\,\phi(x),\,\mathbb{K} + \mathbb{V}\,] \,|\, E,\,1\,\rangle$$

$$= \langle 0 \,|\, [\,\phi(x),\,\mathbb{K} + \mathbb{V}\,] \,|\, E,\,1\,\rangle$$

$$\langle 0 \mid [\phi(x), \mathbb{K}] \mid E, 1 \rangle = -\frac{\hbar^2}{2M} \nabla^2 \langle 0 \mid \phi(x) \mid E, 1 \rangle$$
$$= -\frac{\hbar^2}{2M} \nabla^2 \psi_E(x)$$
$$\langle 0 \mid [\phi(x), \mathbb{V}] \mid E, 1 \rangle = 0$$
$$\rightarrow \quad \left\langle 0 \left| \left[\phi(x), H^{FT} \right] \right| E, 1 \right\rangle = -\frac{\hbar^2}{2M} \nabla^2 \psi_E(x)$$
$$= H_1^{QM} \Psi_E(x)$$

As expected, there're no anyons in a 1-particle system.

2-Particle system

For $N = 2$,

$$\boldsymbol{C}(\boldsymbol{r}_1) = \overline{\alpha} \nabla_1 \theta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad \boldsymbol{C}(\boldsymbol{r}_2) = \overline{\alpha} \nabla_2 \theta(\boldsymbol{r}_2 - \boldsymbol{r}_1)$$

$$\rightarrow \quad \langle 0 | \ \phi(x_1) \Delta \mathrm{K}(x_2) = \frac{\hbar^2}{2M} \left(2 i \frac{\check{q}}{\hbar} \boldsymbol{C}(\boldsymbol{r}_2) \cdot \nabla_2 + \left(\frac{\check{q}}{\hbar} \right)^2 \boldsymbol{C}(\boldsymbol{r}_2)^2 \right) \langle 0 \mid \phi(x_1) \phi(x_2)$$

$$\langle 0 | [\phi(x_2), \mathbb{C}(x_1) \phi(x_1)] = -\left(2 i \frac{\check{q}}{\hbar} \overline{\alpha} \nabla_1 \theta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \nabla_1 + \left(\frac{\check{q}}{\hbar} \right)^2 \overline{\alpha}^2 [\nabla' \theta(\boldsymbol{r}_1 - \boldsymbol{r}_2)]^2 \right) \phi(x_2) \phi(x_1)$$

$$\therefore \quad -\frac{\hbar^2}{2M} \langle 0 | \phi(x_1) \Delta \mathrm{K}(x_2) = \frac{\hbar^2}{2M} \left(2 i \frac{\check{q}}{\hbar} \boldsymbol{C}(\boldsymbol{r}_1) \cdot \nabla_1 + \left(\frac{\check{q}}{\hbar} \right)^2 \boldsymbol{C}(\boldsymbol{r}_1)^2 \right) \langle 0 \mid \phi(x_1) \phi(x_2)$$

Putting everything together, we have

$$\boldsymbol{D}^2 = \nabla^2 - 2 i \frac{\check{q}}{\hbar} \boldsymbol{C} \cdot \nabla - \left(\frac{\check{q}}{\hbar} \right)^2 \boldsymbol{C}^2$$

$$\rightarrow \quad \langle 0 | [\phi(x_1) \phi(x_2), \mathbb{K}] = -\frac{\hbar^2}{2M} (\boldsymbol{D}_1^2 + \boldsymbol{D}_2^2) \langle 0 \mid \phi(x_1) \phi(x_2)$$

$$\langle 0 \mid [\phi(x_1) \phi(x_2), \mathbb{K}] \mid E, 2 \rangle = -\frac{\hbar^2}{2M} (\boldsymbol{D}_1^2 + \boldsymbol{D}_2^2) \Psi_E(x_1, x_2)$$
$$\langle 0 \mid [\phi(x_1) \phi(x_2), \mathbb{V}] \mid E, 2 \rangle = U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \Psi_E(x_1, x_2)$$

$$\rightarrow \quad \left\langle 0 \left| \left[\phi(x_1) \phi(x_2), H^{FT} \right] \right| E, 2 \right\rangle = \left[-\frac{\hbar^2}{2M} (\boldsymbol{D}_1^2 + \boldsymbol{D}_2^2) + U(\boldsymbol{r}_2 - \boldsymbol{r}_1) \right] \Psi_E(x_1, x_2)$$
$$= H_2^{QM} \Psi_E(x_1, x_2)$$

With some labor, one can prove the general N-particle case by the method of induction.