8.5. Anyon Field Operators

In this section, everything is evaluated at the same time t, e.g.,

$$x = (t, \mathbf{r}) \qquad x' = (t, \mathbf{r}') \qquad x_a = (t, \mathbf{r}_a)$$

As in the QM case,

$$\mathbf{D}\,\phi = \left(\nabla - i\,\frac{\check{q}}{\hbar}\,\mathbf{C}\right)\phi = \nabla\,\psi$$

if we set

$$\psi(x) = e^{i\,\alpha\,\Theta(x)}\,\phi(x) \qquad \text{with} \qquad \nabla\Theta = -\frac{\check{q}}{\hbar\alpha}\,\mathbf{C}$$

From 8.4.a._EquivalenceToQMSystem.pdf, we have

$$\mathbf{C}(x) = \frac{\hbar\,\alpha}{\check{q}}\int d^2 r'\,\nabla\,\theta(\mathbf{r}-\mathbf{r}')\,\rho(x') \qquad (\,\rho = \phi^+\,\phi\,)$$

$$\rightarrow \qquad \nabla\Theta = -\int d^2 r'\,\nabla\,\theta(\mathbf{r}-\mathbf{r}')\,\rho(x')$$

$$\therefore \qquad \Theta(x) = -\int d^2 r'\,\theta(\mathbf{r}-\mathbf{r}')\,\rho(x')$$

Note: For stationary states, $\rho(x) = \rho(\mathbf{r})$

$$\rightarrow \qquad \mathbf{C}(x) = \mathbf{C}(\mathbf{r}) \qquad \& \qquad \Theta(x) = \Theta(\mathbf{r})$$

$$[\,\Theta(x),\ \phi(x')\,] = -\int d^2 r_1\,\theta(\mathbf{r}-\mathbf{r}_1)\,[\,\rho(x_1)\,,\ \phi(x')\,]$$

$$[\,\rho(x_1)\,,\ \phi(x')\,] = [\,\phi^+(x_1)\,,\ \phi(x')\,]\,\phi(x_1) = -\delta(\mathbf{r}_1 - \mathbf{r}')\,\phi(x')$$

$$\rightarrow \qquad [\,\Theta(x),\ \phi(x')\,] = \theta(\mathbf{r}-\mathbf{r}')\,\phi(x')$$

Using Hausdorff's formula

$$e^A\,B\,e^{-A} = \sum_{n=0}^{\infty} (\mathrm{ad}_A)^n\,B \qquad (\,\mathrm{ad}_A\,B = [A,\,B]\,)$$

$$= B + [A,\,B] + \frac{1}{2}[A,\,[A,\,B]\,] + \ldots$$

with $A = i\,\alpha\,\Theta(x')$ & $B = \phi(x)$, we have

$$[\,i\,\alpha\,\Theta(x')\,,\ \phi(x)\,] = i\,\alpha\,\theta(\mathbf{r}'-\mathbf{r})\,\phi(x)$$

$$[\,i\,\alpha\,\Theta(x')\,,[\,i\,\alpha\,\Theta\,(x')\,,\phi(x)\,]\,] = (i\,\alpha)^2\,\theta(\mathbf{r}'-\mathbf{r})\,[\,\Theta\,(x')\,,\ \phi(x)\,] = (i\,\alpha)^2\,\theta(\mathbf{r}'-\mathbf{r})^2\,\phi(x)$$

$$(\mathrm{ad}_A)^n\,B = \frac{(i\,\alpha)^2}{n!}\,\theta(\mathbf{r}'-\mathbf{r})^n\,\phi(x)$$

$$\rightarrow \qquad e^{i\,\alpha\,\Theta(x')}\,\phi(x)\,e^{-i\,\alpha\,\Theta(x')} = \sum_{n=0}^{\infty}\frac{(i\,\alpha)^2}{n!}\,\theta(\mathbf{r}'-\mathbf{r})^n\,\phi(x) = e^{i\,\alpha\,\theta(\mathbf{r}'-\mathbf{r})}\,\phi(x)$$

$$\therefore \qquad \psi(x)\,\psi(x') = e^{i\,\alpha\,\Theta(x)}\,\phi(x)\,e^{i\,\alpha\,\Theta(x')}\,\phi(x')$$

$$= e^{i\,\alpha\,\Theta(x)}\,e^{i\,\alpha\,\Theta(x')}\,e^{-i\,\alpha\,\Theta(x')}\,\phi(x)\,e^{i\,\alpha\,\Theta(x')}\,\phi(x')$$

$$= e^{i\,\alpha\,\Theta(x)}\,e^{i\,\alpha\,\Theta(x')}\,e^{-i\,\alpha\,\theta(\mathbf{r}'-\mathbf{r})}\,\phi(x)\,\phi(x')$$

$$= e^{-i\,\alpha\,\theta(\mathbf{r}'-\mathbf{r})}\,e^{i\,\alpha\,\Theta(x)}\,e^{i\,\alpha\,\Theta(x')}\,\phi(x)\,\phi(x') \qquad (\theta \text{ is not an operator })$$

$x \leftrightarrow x'$ gives

$$\psi(x')\,\psi(x) = e^{-i\,\alpha\,\theta(\mathbf{r}-\mathbf{r}')}\,e^{i\,\alpha\,\Theta(x')}\,e^{i\,\alpha\,\Theta(x)}\,\phi(x')\,\phi(x)$$

Using

$$[A\,B,\,C] = A\,[B,\,C] + [A,\,C]\,B \qquad\qquad [A\,,\,B\,C] = [A,\,B]\,C + B\,[A,\,C]$$

we have

$$\begin{aligned}[\,\rho(x)\,,\,\rho(x')\,] &= [\,\phi^+(x)\,\phi(x)\,,\;\phi^+(x')\,\phi(x')\,] \\ &= [\,\phi^+(x)\,,\;\phi^+(x')\,\phi(x')\,]\,\phi(x) + \phi^+(x)\,[\,\phi(x)\,,\;\phi^+(x')\,\phi(x')\,] \\ &= \phi^+(x')\,[\,\phi^+(x)\,,\,\phi(x')\,]\,\phi(x) + \phi^+(x)\,[\,\phi(x)\,,\;\phi^+(x')\,]\,\phi(x') \\ &= -\phi^+(x')\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\phi(x) + \phi^+(x)\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\phi(x') \\ &= 0\end{aligned}$$

$$\to \qquad [\,\Theta(x)\,,\;\Theta(x')\,] = \int d^2 r_1 \int d^2 r_2\,\theta(\boldsymbol{r}-\boldsymbol{r}_1)\,\theta(\boldsymbol{r}'-\boldsymbol{r}_2)\,[\,\rho(x_1,\,\rho(x_2)\,]\;=0$$

Using

$$\theta(-\boldsymbol{r}) = \theta(\boldsymbol{r}) \pm \pi \qquad \text{for} \quad \begin{matrix} 0 \le \theta < 2\pi \\ -\pi < \theta \le \pi \end{matrix}$$

$$0 \le \alpha \le 1$$

we have

$$\psi(x)\,\psi(x') - e^{\pm i\,\alpha\,\pi}\;\psi(x')\,\psi(x) = e^{-i\,\alpha\,\theta(\boldsymbol{r}'-\boldsymbol{r})}\;e^{i\,\alpha\,\Theta(x)}\;e^{i\,\alpha\,\Theta(x')}\,[\,\phi(x),\,\phi(x')\,] = 0$$

$$\to \qquad \psi(x)\,\psi(x') = e^{\pm i\,\alpha\,\pi}\;\psi(x')\,\psi(x)$$

which is just the operator description of an anyon exchange.

Note: The ambiguity in sign is to be expected since x & x' are equivalent so that $e^{\pm i\,\alpha\,\pi}$ can be on either side of the equation.

Setting $x = x'$, we have

$$\psi(x)^2 = e^{\pm i\,\alpha\,\pi}\;\psi(x)^2$$

Since $e^{\pm i\,\alpha\,\pi} \neq 0$, we have

$$\psi(x)^2 = 0 \qquad\qquad \to \qquad \langle 0 \mid \psi(x)^2 = 0$$

i.e., two anyons can't occupy the same space at the same time.

$$\rho^+ = \rho \qquad\qquad \to \qquad \Theta^+ = \Theta$$

$$\therefore \qquad \psi(x)\,\psi^+(x') = e^{i\,\alpha\,\Theta(x)}\;\phi(x)\,\phi^+(x')\,e^{-i\,\alpha\,\Theta(x')}$$

$$\begin{aligned}\psi^+(x')\,\psi(x) &= \phi^+(x')\,e^{-i\,\alpha\,\Theta(x')}\;e^{i\,\alpha\,\Theta(x)}\;\phi(x) \\ &= \phi^+(x')\,e^{i\,\alpha\,\Theta(x)}\;e^{-i\,\alpha\,\Theta(x')}\;\phi(x) \\ &= e^{i\,\alpha\,\Theta(x)}\;e^{-i\,\alpha\,\Theta(x)}\;\phi^+(x')\,e^{i\,\alpha\,\Theta(x)}\;e^{-i\,\alpha\,\Theta(x')}\;\phi(x)\,e^{i\,\alpha\,\Theta(x')}\;e^{-i\,\alpha\,\Theta(x')}\end{aligned}$$

$$\begin{aligned}e^{-i\,\alpha\,\Theta(x)}\;\phi^+(x')\,e^{i\,\alpha\,\Theta(x)} &= \left[\,e^{-i\,\alpha\,\Theta(x)}\;\phi(x')\,e^{i\,\alpha\,\Theta(x)}\;\right]^+ \\ &= \left[\,e^{-i\,\alpha\,\theta(\boldsymbol{r}-\boldsymbol{r}')}\;\phi(x')\,\right]^+ \\ &= e^{i\,\alpha\,\theta(\boldsymbol{r}-\boldsymbol{r}')}\;\phi^+(x')\end{aligned}$$

$$\begin{aligned}\to \qquad \psi^+(x')\,\psi(x) &= e^{i\,\alpha\,\Theta(x)}\;e^{i\,\alpha\,\theta(\boldsymbol{r}-\boldsymbol{r}')}\;\phi^+(x')\,e^{-i\,\alpha\,\theta(\boldsymbol{r}'-\boldsymbol{r})}\;\phi(x)\,e^{-i\,\alpha\,\Theta(x')} \\ &= e^{i\,\alpha\,\{\,\theta(\boldsymbol{r}-\boldsymbol{r}')-\theta(\boldsymbol{r}'-\boldsymbol{r})\,\}}\;e^{i\,\alpha\,\Theta(x)}\;\phi^+(x')\,\phi(x)\,e^{-i\,\alpha\,\Theta(x')} \\ &= e^{\mp i\,\alpha\,\pi}\;e^{i\,\alpha\,\Theta(x)}\;\phi^+(x')\,\phi(x)\,e^{-i\,\alpha\,\Theta(x')}\end{aligned}$$

$$\begin{aligned}\therefore \qquad \psi(x)\,\psi^+(x') - e^{\pm i\,\alpha\,\pi}\;\psi^+(x')\,\psi(x) &= e^{i\,\alpha\,\Theta(x)}\,[\,\phi(x),\,\phi^+(x')\,]\,e^{-i\,\alpha\,\Theta(x')} \\ &= \delta(\boldsymbol{r}-\boldsymbol{r}')\end{aligned}$$

If we define the anyonic commutator by

$$[\,A,\,B\,]_\alpha = A\,B - e^{\pm i\,\alpha\,\pi}\,B\,A$$

then we have

$$[\,\psi(x),\;\psi(x')\,]_\alpha = 0$$

$$[\,\psi(x),\ \psi^{+}(x')\,]_{\alpha} = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

where $\alpha = 0$ & 1 corresponds to the usual commutator & anti-commutator, respectively.