10.1. Planar Electrons

Consider a quasi-2-D system, e.g., electrons at the interface of semiconductor heterostructures.

Wave function for 1-particle is of the form

$$\Psi(x, y, z) = \psi(\boldsymbol{r})\,\psi_0(z) \qquad \boldsymbol{r} = (x, y)$$

Delta Function Potential

Let $\quad V(z) = -V_0\,\delta(z)$

$$\rightarrow \quad \left(-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V_0\,\delta(z)\right)\psi_0(z) = E\,\psi_0(z)$$

For $z \neq 0$,

$$-\frac{\hbar^2}{2M}\frac{d^2}{dz^2}\psi_0 = E\,\psi_0$$

Set $\quad \beta = \dfrac{\sqrt{2ME}}{\hbar}$

$$\rightarrow \quad \psi_0'' = -\beta^2\,\psi_0$$

$$\therefore \quad \psi_0 = C\,e^{-\beta|z|}$$

$$\int_{-\infty}^{\infty} dz\,\psi_0^2 = 2C^2\int_0^{\infty} dz\,e^{-2\beta z} = \frac{C^2}{\beta} = 1$$

$$\rightarrow \quad C = \sqrt{\beta}$$

$$\& \quad \psi_0 = \sqrt{\beta}\,e^{-\beta|z|} \qquad \forall\ z \neq 0$$

$\int_{-\infty}^{\infty} dz$ both sides of the Schrodinger eq. gives

$$\int_{-\infty}^{\infty} dz\left(-\frac{d^2}{dz^2} + v_0\,\delta(z)\right)\psi_0 = \beta^2\int_{-\infty}^{\infty} dz\,\psi_0 \qquad \left(v_0 = \frac{2MV_0}{\hbar^2}\right)$$

$$\int_{-\infty}^{\infty} dz\,\psi_0 = 2\sqrt{\beta}\int_0^{\infty} dz\,e^{-\beta z} = \frac{2}{\sqrt{\beta}}$$

$$\rightarrow \quad -\psi_0'(\infty) + \psi_0'(-\infty) + v_0\,\psi_0(0) = 2\beta^{3/2}$$

$$\psi_0'(z) = \begin{cases} -\beta^{3/2}\,e^{-\beta z} & z > 0 \\ \beta^{3/2}\,e^{\beta z} & z < 0 \end{cases} \quad \rightarrow \quad \psi_0'(\pm\infty) = 0$$

$$\therefore \quad \psi_0(0) = \frac{2}{v_0}\beta^{3/2}$$

Continuity of $\psi_0 \quad \rightarrow \quad \sqrt{\beta} = \dfrac{2}{v_0}\beta^{3/2}$

$$\therefore \quad \beta = \frac{1}{2}v_0 = \frac{MV_0}{\hbar^2}$$

Note that from the normalization equation, we see that

$$\lim_{\beta\to\infty}\psi_0^2 = \lim_{\beta\to\infty}\beta\,e^{-2\beta|z|} = \delta(z)$$

$\therefore \qquad \lim_{\beta \to \infty} \psi_0 = \sqrt{\delta(z)}$

& the system becomes truely 2-D.

In which case,

$$\rho(\boldsymbol{r}, z) = [\psi(\boldsymbol{r})\, \psi_0(z)]^* \, \psi(\boldsymbol{r})\, \psi_0(z) = \rho(\boldsymbol{r})\, \delta(z)$$

This formula is easily generalized to an N-particle system by setting

$$\rho(\boldsymbol{r}) = |\, \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)\, |^2 = \sum_{a=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_a)$$

For particles truly confined to a plane, the Coulomb energy reduces to

$$\mathbb{V}_C = \frac{1}{2} \frac{e^2}{4\pi\varepsilon} \int d^2 r \int d^2 r' \, \frac{\rho(\boldsymbol{r})\, \rho(\boldsymbol{r}')}{|\, \boldsymbol{r} - \boldsymbol{r}'\, |}$$

Note: We use e to denote the charge of the particle because q will be used to denote the magnitude of the wave vector $\boldsymbol{q}$. If the particle is an electron, then $e < 0$.

In contrast, $e_{\text{Ezawa}} = |\, e\, |$ for electrons.

Fourier transform of the 2-D Coulomb potential

$$V(\boldsymbol{r}) = \frac{e^2}{4\pi\varepsilon r}$$

is

$$V(\boldsymbol{q}) = \int d^2 r\, V(\boldsymbol{r})\, e^{-i\,\boldsymbol{q}\cdot\boldsymbol{r}} = \frac{e^2}{4\pi\varepsilon} \int d^2 r \, \frac{e^{-i\,\boldsymbol{q}\cdot\boldsymbol{r}}}{r}$$

Note: Ezawa multiplies the integral by $(2\pi)^{-d/2}$, where d is the dimension.

Without lost of generality, we can set the x-axis along $-\boldsymbol{q}$ so that

$$\int d^2 r \, \frac{e^{-i\,\boldsymbol{q}\cdot\boldsymbol{r}}}{r} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, e^{i\,q\,r\cos\theta}$$

$$= 2\pi \int_0^{\infty} dr\, J_0(q\,r) \qquad\qquad (\, J_\nu = \text{Bessel function}\,)$$

From M.Abramowitz, “Handbook of Mathematical Functions”, Formula 11.4.17,

$$\int_0^{\infty} dr\, J_\nu(r) = 1 \qquad \forall \ \text{Re}\, \nu > -1$$

$$\rightarrow \qquad V(\boldsymbol{q}) = \frac{e^2}{4\pi\varepsilon} \frac{2\pi}{q} = \frac{e^2}{2\varepsilon q}$$

In a QH system, the particles in the $z = 0$ plane are electrons supplied by a δ-doped donor layer at $z = d$. As far as the electron dynamics are concerned, the ionized donors can be modeled by a layer of uniform charge density $-e\,\rho_0$. The total charge density is then

$$\rho_e(\boldsymbol{r}, z) = e\,[\, \rho(\boldsymbol{r})\, \delta(z) - \rho_0\, \delta(z - d)\,]$$

where $e < 0$ is the charge of the electron &

$$\int d^3 x\, \rho_e(\boldsymbol{r}, z) = 0$$

Obviously, the same relations apply to holes (with $e > 0$) in the $z = 0$ plane supplied by a δ-doped acceptor layer at $z = d$.

Let $\quad \Delta\rho(\boldsymbol{r}) = \rho(\boldsymbol{r}) - \rho_0$

$$\rightarrow \qquad \rho_e(\boldsymbol{r}, z) = e\,[\, \Delta\rho(\boldsymbol{r})\, \delta(z) + \rho_0\, \delta(z) - \rho_0\, \delta(z - d)\,]$$

The Coulomb energy becomes

$$\mathbb{V}_C = \frac{1}{2} \int d^3 x \int d^3 x' \, \frac{\rho_e(\boldsymbol{r}, z)\, \rho_e(\boldsymbol{r}', z')}{4\pi\varepsilon \sqrt{(\boldsymbol{r} - \boldsymbol{r}')^2 + (z - z')^2}}$$

$$= \frac{1}{2}\frac{e^2}{4\pi\varepsilon}\int d^3x\int d^2r'\,[\Delta\rho(\boldsymbol{r})\,\delta(z)+\rho_0\,\delta(z)-\rho_0\,\delta(z-d)]$$

$$*\left[\frac{\Delta\rho(\boldsymbol{r}')}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+z^2}}+\frac{\rho_0}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+z^2}}-\frac{\rho_0}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+(z-d)^2}}\right]$$

$$= \frac{1}{2}\frac{e^2}{4\pi\varepsilon}\int d^2r\int d^2r'\left\{\Delta\rho(\boldsymbol{r})\left[\frac{\Delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}+\frac{\rho_0}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{\rho_0}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}\right]\right.$$

$$+\rho_0\left[\frac{\Delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}+\frac{\rho_0}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{\rho_0}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}\right]$$

$$\left.-\rho_0\left[\frac{\Delta\rho(\boldsymbol{r}')}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}+\frac{\rho_0}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}-\frac{\rho_0}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\right\}$$

$$= \frac{1}{2}\frac{e^2}{4\pi\varepsilon}\int d^2r\int d^2r'\left\{\frac{\Delta\rho(\boldsymbol{r})\,\Delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}+\rho_0[\Delta\rho(\boldsymbol{r})+\Delta\rho(\boldsymbol{r}')]\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}\right)\right.$$

$$\left.+2\rho_0^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}\right)\right\}$$

Each integral term is invariant under $\boldsymbol{r} \leftrightarrow \boldsymbol{r}'$.

→

$$\mathbb{V}_C = \frac{1}{2}\frac{e^2}{4\pi\varepsilon}\int d^2r\int d^2r'\left\{\frac{\Delta\rho(\boldsymbol{r})\,\Delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}+2\rho_0[\Delta\rho(\boldsymbol{r})+\rho_0]\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}}\right)\right\}$$

Let $R = |\boldsymbol{r}-\boldsymbol{r}'|$

→ $$\int d^2r'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 2\pi\int_0^\infty dR = 2\pi\lim_{R\to\infty}R$$

$$\int d^2r'\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}} = 2\pi\int_0^\infty dR\,\frac{R}{\sqrt{R^2+d^2}} = 2\pi\sqrt{R^2+d^2}\,\Big|_0^\infty = 2\pi\lim_{R\to\infty}R$$

∴ $$\int d^2r'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\int d^2r'\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+d^2}} = 0$$

→ $$\mathbb{V}_C = \frac{1}{2}\frac{e^2}{4\pi\varepsilon}\int d^2r\int d^2r'\,\frac{\Delta\rho(\boldsymbol{r})\,\Delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

If there are N particles in the plane, the total Hamiltonian may be written as

$$H = \mathbb{K}+\mathbb{V}_C+\mathbb{V}_Z$$

where

$$\mathbb{K} = -\frac{\hbar^2}{2M}\sum_{a=1}^{N}\boldsymbol{D}_a^2 \qquad \boldsymbol{D} = \nabla - i\,\frac{e}{\hbar c}\,\boldsymbol{A}$$

with A being the EM vector potential. The Zeeman term is

$$\mathbb{V}_Z = -\frac{1}{2}\Delta_Z\int d^2r\,[\rho_\uparrow(\boldsymbol{r})-\rho_\downarrow(\boldsymbol{r})]$$

where $\Delta_Z = g\,\mu_B\,B$ is the Zeeman gap & $\rho_\uparrow$ ($\rho_\downarrow$) is the density of spin ↑(↓) particles.

2-D Free Electrons

Consider a system of N free 2-D particles,

$$H = -\frac{\hbar^2}{2M} \sum_{a=1}^{N} \nabla_a^2$$

The 1-particle wave function is

$$\psi(\boldsymbol{r}) = \frac{1}{L} e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} \qquad\qquad \boldsymbol{r} = (x, y)$$

with the box normalization

$$\int d^2 r\, \psi^*(\boldsymbol{r})\, \psi(\boldsymbol{r}) = \int_0^L d x \int_0^L d y\, \psi^*(\boldsymbol{r})\, \psi(\boldsymbol{r}) = 1$$

Its energy is

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 \boldsymbol{k}^2}{2M}$$

Imposing the periodic boundary condition

$$\psi(\boldsymbol{r} + L\,\hat{\boldsymbol{x}}) = \psi(\boldsymbol{r}) = \psi(\boldsymbol{r} + L\,\hat{\boldsymbol{y}})$$

we have

$$\boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y) \qquad\qquad n_i = 0, \pm 1, \pm 2, \ \ldots$$

Density of states in the k-space is therefore

$$D(\boldsymbol{k}) = \left(\frac{L}{2\pi}\right)^2$$

For $L \to \infty$, $\boldsymbol{k}$ becomes a continuous variable.

The energy density of states is then given by

$$\int d\varepsilon\, D(\varepsilon) = \int d^2 k\, D(\boldsymbol{k}) = \left(\frac{L}{2\pi}\right)^2 2\pi \int d k\, k$$

$$\varepsilon = \frac{\hbar^2 k^2}{2M} \qquad \to \qquad d\varepsilon = \frac{\hbar^2}{M} k\, d k$$

$$\therefore \qquad D(\varepsilon) = 2\pi \left(\frac{L}{2\pi}\right)^2 \frac{M}{\hbar^2} \qquad\qquad (L \to \infty)$$

$$= L^2 \frac{M}{2\pi\hbar^2}$$

For fermions, each state can accommodate only 1 particle so that in the ground state,

$$\int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon) = N$$

where the highest occupied energy ε_F (chemical potential at $T = 0$) is usually called the Fermi energy .

$$\to \qquad \varepsilon_F = \frac{N}{L^2} \frac{2\pi\hbar^2}{M} = \frac{2\pi\hbar^2}{M} \rho_0$$

where $\rho_0 = \frac{N}{L^2}$ is the average electron density.

The highest occupied wave vector is

$$k_F = \sqrt{\frac{2 M \varepsilon_F}{\hbar^2}} = \sqrt{4 \pi \rho_0}$$

Thus, in the ground state, the particles occupy a disk of radius k_F centered at the origin in k-space.

For spin 1/2 fermions in the absence of magnetic fields, each state can accommodate at most 2 particles.

→ $D(\varepsilon) = L^2 \frac{M}{\pi \hbar^2}$ $\varepsilon_F = \frac{\pi \hbar^2}{M} \rho_0$ $k_F = \sqrt{2 \pi \rho_0}$