10.2. Cyclotron Motion

General Consideration

For particles in a magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$,

$$H = \frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 = -\frac{\hbar^2}{2M}\boldsymbol{D}^2 \qquad \boldsymbol{D} = \nabla - i\frac{e}{\hbar c}\boldsymbol{A}$$

$\boldsymbol{D}$ is called the covariant derivative.

$$\begin{aligned}[D_i, D_j] &= \left[\partial_i - i\frac{e}{\hbar c}A_i,\ \partial_j - i\frac{e}{\hbar c}A_j\right] \\ &= -i\frac{e}{\hbar c}\left([\partial_i, A_j] + [A_i, \partial_j]\right) \\ &= -i\frac{e}{\hbar c}\left(\partial_i A_j - \partial_j A_i\right) \\ &= -i\frac{e}{\hbar c}\varepsilon_{ijk}B_k\end{aligned}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{1}{i\hbar}[\boldsymbol{r}, H]$$

$$[r_i, D_j] = \left[r_i, \partial_j - i\frac{e}{\hbar c}A_j\right] = [r_i, \partial_j] = -\delta_{ij}$$

$$\rightarrow \qquad [r_i, H] = -\frac{\hbar^2}{2M}[r_i, D_j D_j] = -\frac{\hbar^2}{2M}\left([r_i, D_j]D_j + D_j[r_i, D_j]\right)$$

$$= \frac{\hbar^2}{M}D_i$$

$$\therefore \qquad M\boldsymbol{v} = \frac{\hbar}{i}\boldsymbol{D} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$$

in contrast with

$$\boldsymbol{p} = \frac{\hbar}{i}\nabla$$

Note that the same result can be obtained by

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{1}{M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) = \frac{1}{M}\frac{\hbar}{i}\boldsymbol{D}$$

$\boldsymbol{v}$ is hermitian since $\boldsymbol{p}$ & $\boldsymbol{A}$ are:

$$\boldsymbol{v} = \frac{1}{M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) \qquad \rightarrow \qquad \boldsymbol{v}^+ = \boldsymbol{v}$$

Alternatively, $\boldsymbol{v} = \frac{\hbar}{Mi}\boldsymbol{D}$

$$\rightarrow \qquad \boldsymbol{v}^+ = -\frac{\hbar}{Mi}\overleftarrow{\boldsymbol{D}}^* = -\frac{\hbar}{Mi}\left(\overleftarrow{\nabla} - i\frac{e\hbar}{c}\boldsymbol{A}\right)^* = -\frac{\hbar}{Mi}\left(-\nabla + i\frac{e\hbar}{c}\boldsymbol{A}\right) = \frac{\hbar}{Mi}\boldsymbol{D} = \boldsymbol{v}$$

In terms of $\boldsymbol{v}$, we have

$$H = \frac{1}{2}M\boldsymbol{v}^2$$

$$[r_i, v_j] = \frac{\hbar}{M i}[r_i, D_j] = i\,\frac{\hbar}{M}\,\delta_{i j}$$

$$[v_i, v_j] = -\frac{\hbar^2}{M^2}[D_i, D_j] = i\,\frac{e\,\hbar}{M^2 c}\,\varepsilon_{i j k}\,B_k$$

$$\begin{aligned}[v_i\,,\,\boldsymbol{v}^2] &= [v_i\,,\,v_j]\,v_j + v_j\,[v_i\,,\,v_j] \\ &= i\,\frac{e\,\hbar}{M^2 c}\,\varepsilon_{i j k}\,(B_k\,v_j + v_j\,B_k) \\ &= -i\,\frac{e\,\hbar}{M^2 c}\,\varepsilon_{i j k}\,(B_j\,v_k - v_j\,B_k)\end{aligned}$$

$$\begin{aligned}(B_j\,v_k - v_j\,B_k)\,f &= \frac{\hbar}{M i}\left\{B_j\left(\partial_k - i\,\frac{e\,\hbar}{c}\,A_k\right)f - \left(\partial_j - i\,\frac{e\,\hbar}{c}\,A_j\right)(B_k\,f)\right\} \\ &= \frac{\hbar}{M i}\left\{B_j\,\partial_k f - \partial_j(B_k\,f) - i\,\frac{e\,\hbar}{c}\,(B_j\,A_k - A_j\,B_k)\,f\right\} \\ &= \frac{\hbar}{M i}\left\{B_j\,\partial_k f - B_k\,\partial_j f - (\partial_j B_k)\,f - i\,\frac{e\,\hbar}{c}\,(B_j\,A_k - A_j\,B_k)\,f\right\} \\ &= \frac{\hbar}{M i}\left\{B_j\left(\partial_k - i\,\frac{e\,\hbar}{c}\,A_k\right)f - B_k\left(\partial_j - i\,\frac{e\,\hbar}{c}\,A_j\right)f - (\partial_j B_k)\,f\right\}\end{aligned}$$

$$\begin{aligned}\rightarrow \qquad \varepsilon_{i j k}\,(B_j\,v_k - v_j\,B_k)\,f &= \frac{\hbar}{M i}\left\{2\,\varepsilon_{i j k}\,B_j\left(\partial_k - i\,\frac{e\,\hbar}{c}\,A_k\right)f - (\varepsilon_{i j k}\,\partial_j B_k)\,f\right\} \\ &= \frac{\hbar}{M i}\,\{2\,\boldsymbol{B}\times\boldsymbol{D} - (\nabla\times\boldsymbol{B})\}_i\,f \\ &= \left\{2\,\boldsymbol{B}\times\boldsymbol{v} - \frac{\hbar}{M i}\,(\nabla\times\boldsymbol{B})\right\}_i f\end{aligned}$$

$$\begin{aligned}[\boldsymbol{v}\,,\,\boldsymbol{v}^2] &= -i\,\frac{e\,\hbar}{M^2 c}\,(\boldsymbol{B}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{B}) \\ &= -i\,\frac{e\,\hbar}{M^2 c}\left(2\,\boldsymbol{B}\times\boldsymbol{v} - \frac{\hbar}{M i}\,(\nabla\times\boldsymbol{B})\right)\end{aligned}$$

For a uniform $\boldsymbol{B}$,

$$[\boldsymbol{v}\,,\,\boldsymbol{v}^2] = -2\,i\,\frac{e\,\hbar}{M^2 c}\,\boldsymbol{B}\times\boldsymbol{v} = -2\,i\,\frac{\hbar\,\mathbb{e}\,\omega_c}{M}\,\hat{\boldsymbol{B}}\times\boldsymbol{v}$$

where

$$\omega_c = \frac{|\,e\,|\;B}{M\,c} = \text{cyclotron frequency}$$

$$\mathbb{e} = \frac{e}{|\,e\,|} = \text{sign of } e$$

$$[\boldsymbol{v}\,,\,H] = \frac{1}{2}\,M\,[\boldsymbol{v}\,,\,\boldsymbol{v}^2] = -i\,\frac{e\,\hbar}{M\,c}\,\boldsymbol{B}\times\boldsymbol{v} = -i\,\hbar\,\mathbb{e}\,\omega_c\,\hat{\boldsymbol{B}}\times\boldsymbol{v}$$

$$\therefore \qquad \dot{\boldsymbol{v}} = \frac{1}{i\,\hbar}[\boldsymbol{v}, H] = -\frac{e}{M\,c}\,\boldsymbol{B}\times\boldsymbol{v} \qquad \text{(Lorentz force)}$$

$$= -\mathbb{e}\,\omega_c\,\hat{\boldsymbol{B}}\times\boldsymbol{v} \qquad \text{(cyclotron eq.)}$$

Classically, this means for $\mathbb{e} = +1$, the particle goes clockwise in a circular orbit about $\hat{\boldsymbol{B}}$ (left hand rule with thumb // $\boldsymbol{B}$).

Planar Particles

For planar particles in a uniform magnetic field $\boldsymbol{B} = (0, 0, B)$,

$$[D_i, D_j] = -i \frac{e B}{\hbar c} \varepsilon_{i j 3} = -i \frac{M \mathrm{e} \omega_c}{\hbar} \varepsilon_{i j 3} = -i \frac{\mathrm{e}}{l_B^2} \varepsilon_{i j 3}$$

where

$$l_B^2 = \frac{\hbar c}{|e| B} \qquad \omega_c = \frac{|e| B}{M c} \qquad \frac{\hbar}{M} = l_B^2 \omega_c$$

$$\rightarrow \quad [v_i, v_j] = i \frac{e \hbar B}{M^2 c} \varepsilon_{i j 3} = i\, \mathrm{e} \frac{\hbar \omega_c}{M} \varepsilon_{i j 3} = i\, \mathrm{e}\, (l_B \omega_c)^2 \varepsilon_{i j 3}$$

$$H = \frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 = \frac{1}{2} M \boldsymbol{v}^2$$

$$= \frac{1}{2} M \left(v_1^2 + v_2^2\right)$$

Analogous to the harmonic oscillator, we set

$$a = \alpha\,(v_1 + i\,\mathrm{e}\, v_2)$$

where α is a normalization constant.

$$\rightarrow \quad a^+ = \alpha^* (v_1 - i\,\mathrm{e}\, v_2)$$

$$\therefore \quad a\, a^+ = \alpha^* \alpha \left(\boldsymbol{v}^2 - i\,\mathrm{e}\,[v_1, v_2]\right)$$

$$= \alpha^* \alpha \left(\boldsymbol{v}^2 + \frac{\hbar \omega_c}{M}\right)$$

$$a^+ a = \alpha^* \alpha \left(\boldsymbol{v}^2 + i\,\mathrm{e}\,[v_1, v_2]\right)$$

$$= \alpha^* \alpha \left(\boldsymbol{v}^2 - \frac{\hbar \omega_c}{M}\right)$$

$$\rightarrow \quad [a, a^+] = 2 \alpha^* \alpha \frac{\hbar \omega_c}{M} \equiv 1$$

$$\therefore \quad \alpha = \sqrt{\frac{M}{2 \hbar \omega_c}} = \frac{1}{\sqrt{2}\, l_B \omega_c}$$

$$\rightarrow \quad \boldsymbol{v}^2 = \frac{1}{2 \alpha^* \alpha}\left(a\, a^+ + a^+ a\right) = \frac{\hbar \omega_c}{M}\left(a\, a^+ + a^+ a\right) = (l_B \omega_c)^2 \left(a\, a^+ + a^+ a\right)$$

$$H = \frac{1}{2} \hbar \omega_c \left(a\, a^+ + a^+ a\right) = \hbar \omega_c \left(a^+ a + \frac{1}{2}\right)$$

&

$$a = \sqrt{\frac{M}{2 \hbar \omega_c}}\,(v_1 + i\,\mathrm{e}\, v_2) = \frac{1}{\sqrt{2}\, l_B \omega_c}(v_1 + i\,\mathrm{e}\, v_2)$$

$$a^+ = \sqrt{\frac{M}{2 \hbar \omega_c}}\,(v_1 - i\,\mathrm{e}\, v_2) = \frac{1}{\sqrt{2}\, l_B \omega_c}(v_1 - i\,\mathrm{e}\, v_2)$$

Thus, energy of the system is quantized & given by

$$E_n = \hbar \omega_c \left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots$$

R

In classical physics, the cyclotron orbit about the center $\boldsymbol{R}$ is given by (orbit is clockwise in xy-plane for $e = +1$ & $\boldsymbol{B} = B\hat{\boldsymbol{z}}$)

$$\boldsymbol{r}_\perp = \boldsymbol{R}_\perp + \frac{c\,M}{e\,B}\hat{\boldsymbol{B}}\times\boldsymbol{v} = \boldsymbol{R}_\perp + \frac{e}{\omega_c}\hat{\boldsymbol{B}}\times\boldsymbol{v}$$

where $\perp$ denotes the component perpendicular to $\boldsymbol{B}$. Radius of the orbit is

$$|\,\boldsymbol{r}_\perp - \boldsymbol{R}_\perp\,| = \left|\frac{e}{\omega_c}\hat{\boldsymbol{B}}\times\boldsymbol{v}\right| = \frac{v_\perp}{\omega_c}$$

Thus, $\boldsymbol{R}_\perp = \boldsymbol{r}_\perp - \frac{e}{\omega_c}\hat{\boldsymbol{z}}\times\boldsymbol{v}$ is the center if the particle is in a cyclotron orbit.

Keeping the same definition in the quantum case with $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ & planar particles, we have

$$\boldsymbol{R} = \boldsymbol{r} - \frac{e}{\omega_c}\hat{\boldsymbol{z}}\times\boldsymbol{v} \qquad \text{or} \qquad R_i = r_i + \frac{e}{\omega_c}\varepsilon_{ij3}\,v_j$$

Caution: for electrons, $\boldsymbol{R} = (R_1, R_2) = (X, Y)_{\text{Ezawa}} = \boldsymbol{X}_{\text{Ezawa}}$ if $e_{\text{Ezawa}} = e$.

However, $M\,\boldsymbol{v} = \boldsymbol{P}_{\text{Ezawa}}$ only if $e_{\text{Ezawa}} = -e$.

$[v_i, v_j] \neq 0$ means that the components of $\boldsymbol{R}$ (or $\boldsymbol{\chi} = \boldsymbol{r} - \boldsymbol{R}$) do not commute. Using

$$[r_i, v_j] = i\frac{\hbar}{M}\delta_{ij}$$

$$[v_i, v_j] = i\,e\frac{\hbar\omega_c}{M}\varepsilon_{ij3}$$

we have

$$[r_i, R_j] = \left[r_i,\ r_j + \frac{e}{\omega_c}\varepsilon_{jm3}v_m\right]$$
$$= i\frac{\hbar e}{M\omega_c}\varepsilon_{jm3}\delta_{im}$$
$$= -i\frac{\hbar e}{M\omega_c}\varepsilon_{ij3} = -i\frac{\hbar c}{eB}\varepsilon_{ij3} = -i\,e\,l_B^2\,\varepsilon_{ij3}$$

$$[v_i, R_j] = \left[v_i,\ r_j + \frac{e}{\omega_c}\varepsilon_{jm3}v_m\right]$$
$$= -i\frac{\hbar}{M}\delta_{ij} + i\frac{\hbar}{M}\varepsilon_{jm3}\varepsilon_{im3}$$
$$= 0$$

$$\therefore \quad [R_i, R_j] = \left[r_i + \frac{e}{\omega_c}\varepsilon_{ik3}v_k,\ R_j\right]$$
$$= [r_i, R_j]$$
$$= -i\frac{\hbar e}{M\omega_c}\varepsilon_{ij3} = -i\frac{\hbar c}{eB}\varepsilon_{ij3} = -i\,e\,l_B^2\,\varepsilon_{ij3}$$

$$[v_i, R_j] = 0$$

$$\rightarrow \quad [R_i, H] = \left[R_i,\ \frac{1}{2}M\,v_j\,v_j\right] = 0$$

$$\therefore \quad \dot{\boldsymbol{R}} = 0$$

i.e., the center of orbit is stationary, as expected for planar particles.

Let

$$b = \frac{1}{\sqrt{2}\, l_B} (R_1 - i\, e\, R_2) \qquad\qquad b^+ = \frac{1}{\sqrt{2}\, l_B} (R_1 + i\, e\, R_2)$$

$$\to \qquad b\, b^+ = \frac{1}{2\, l_B^2} \left(R_1^2 + R_2^2 + i\, e\, [R_1, R_2] \right)$$

$$b^+\, b = \frac{1}{2\, l_B^2} \left(R_1^2 + R_2^2 - i\, e\, [R_1, R_2] \right)$$

$$\therefore \qquad [\, b, b^+\,] = \frac{1}{l_B^2}\, i\, e\, [R_1, R_2] = 1$$

$$b\, b^+ + b^+\, b = \frac{1}{l_B^2} \boldsymbol{R}^2$$

$$\to \qquad \boldsymbol{R}^2 = 2\, l_B^2 \left(m + \frac{1}{2} \right) \qquad\qquad m = 0, 1, 2, \ldots$$

χ

The particle position relative to the orbit (or guiding) center is

$$\boldsymbol{\chi} = \boldsymbol{r} - \boldsymbol{R} = \frac{e}{\omega_c} \hat{\boldsymbol{z}} \times \boldsymbol{v} = \boldsymbol{R}_{\text{Ezawa}} = \frac{e}{\omega_c} (-v_2, v_1)$$

$$\chi_i = -\frac{e}{\omega_c} \varepsilon_{i j 3}\, v_j$$

$$\to \qquad v_i = e\, \omega_c\, \varepsilon_{i j 3}\, \chi_j \qquad\qquad \boldsymbol{v} = -e\, \omega_c\, \hat{\boldsymbol{z}} \times \boldsymbol{\chi}$$

$$\boldsymbol{v}^+ = \boldsymbol{v} \qquad \to \qquad \boldsymbol{\chi}^+ = \boldsymbol{\chi}$$

$$\left[v_i, v_j \right] = i\, e\, \frac{\hbar \omega_c}{M} \varepsilon_{i j 3}$$

$$\to \qquad [\, \chi_i, H\,] = -\frac{e}{\omega_c} \varepsilon_{i j 3} \left[v_j\,,\, \frac{1}{2} M\, v_k\, v_k \right]$$

$$= -i\, \hbar\, \frac{1}{2} \varepsilon_{i j 3} \left(v_k\, \varepsilon_{j k 3} + \varepsilon_{j k 3}\, v_k \right)$$

$$= i\, \hbar\, v_i$$

$$\therefore \qquad \dot{\boldsymbol{\chi}} = \boldsymbol{v}$$

$$\boldsymbol{v} = -e\, \omega_c\, \hat{\boldsymbol{z}} \times \boldsymbol{\chi} = -e\, \omega_c\, \chi\, \hat{\boldsymbol{\theta}}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector for the polar angle θ. Thus, for $e = +1$, the particle goes clockwise with angular frequency ω_c in a circle centered at $\boldsymbol{R}$ and of radius χ.

Alternatively,

$$\boldsymbol{v} = -e\, \omega_c\, \hat{\boldsymbol{z}} \times \boldsymbol{\chi}$$

$$\to \qquad \dot{\boldsymbol{\chi}} = -e\, \omega_c\, \hat{\boldsymbol{z}} \times \boldsymbol{\chi}$$

$$\therefore \qquad \dot{\chi}_1 = e\, \omega_c\, \chi_2 \qquad\qquad \dot{\chi}_2 = -e\, \omega_c\, \chi_1$$

Let $\qquad z = \chi_1 + i\, \chi_2$

$$\to \qquad \dot{z} = -i\, e\, \omega_c\, z$$

$$z = \left| z \right| e^{-i e \omega_c t}$$

Thus, z goes, for $e = +1$, clockwise in a circle of radius $|z|$.
Treating the complex plane as the $\chi_1 \chi_2$-plane, $\boldsymbol{\chi}$ goes likewise but with radius χ.

In component form,

$$\chi_1 + i \chi_2 = \chi [\cos(\omega_c t) - i e \sin(\omega_c t)]$$

i.e.

$$\chi_1 = \chi \cos(\omega_c t)$$
$$\chi_2 = -e \chi \sin(\omega_c t)$$

$$\chi_i = -\frac{e}{\omega_c} \varepsilon_{ij3} v_j \qquad [v_i, v_j] = i\, e \frac{\hbar\omega_c}{M} \varepsilon_{ij3}$$

$$\to \quad [\chi_i, \chi_j] = \frac{1}{\omega_c^2} \varepsilon_{ik3} \varepsilon_{jm3} [v_k, v_m]$$
$$= i\, e \frac{\hbar}{\omega_c M} \varepsilon_{ik3} \varepsilon_{jm3} \varepsilon_{km3} = i\, e \frac{\hbar}{\omega_c M} \varepsilon_{ik3} \delta_{jk}$$
$$= i\, e \frac{\hbar}{\omega_c M} \varepsilon_{ij3} = i \frac{\hbar c}{e B} \varepsilon_{ij3} = i\, e\, l_B^2 \varepsilon_{ij3}$$
$$= -[R_i, R_j]$$

Comparing with

$$[x, p] = i \hbar \qquad \to \qquad \text{phase space volume of 1 state} = h = 2\pi \; | [x, p] |$$

we see that

$$\text{Area of smallest orbit} = | 2\pi [\chi_1, \chi_2] |$$
$$= \frac{2\pi \hbar c}{|e| B} = \frac{\Phi_0}{B} = 2\pi l_B^2$$

which agrees with the flux quantization discussed previously.

$$\boldsymbol{\chi} = \frac{e}{\omega_c} (-v_2, v_1)$$

$$\to \quad H = \frac{1}{2} M \boldsymbol{v}^2 = \frac{1}{2} M \omega_c^2 \boldsymbol{\chi}^2 = \frac{\hbar \omega_c}{2 l_B^2} \boldsymbol{\chi}^2 = \left(a^+ a + \frac{1}{2}\right) \hbar \omega_c$$

$$\therefore \quad \boldsymbol{\chi}^2 = 2 l_B^2 \left(a^+ a + \frac{1}{2}\right)$$

The cyclotron radius is therefore roughly

$$R_c \simeq \sqrt{\langle \boldsymbol{\chi}^2 \rangle} = l_B \sqrt{2n + 1}$$

For the ground state ($n = 0$),

$$R_c = l_B$$

For orbits centered on the same $\boldsymbol{R}$:

All orbits have the same frequency ω_c.
Higher n means

higher energy $E = \hbar \omega_c \left(n + \frac{1}{2}\right)$,

higher velocity $v = \sqrt{\frac{2E}{M}}$,

& hence larger radius $\chi = \frac{v}{\omega_c}$.

Values of χ^2 is quantized with $\Delta\chi^2 = 2\, l_B^2$.

→ Area S of orbit $\pi\,\chi^2$ is quantized with $\Delta S = 2\,\pi\, l_B^2$.

→ Flux is quantized in units of $2\,\pi\, l_B^2\, B = \Phi_0$.

Eigenstates

From

$$E_n = \hbar\,\omega_c\left(n + \frac{1}{2}\right)$$

$$\dot{\boldsymbol{R}} = 0 \qquad \& \qquad \boldsymbol{R}_m^2 = l_B^2\,(2\,m+1)$$

we see that the eigenstates can be labeled as $|\,n\,m\rangle$.

For the n^{th} Landau level, the cyclotron orbit has a radius

$$R_{c\,n} = l_B\sqrt{2\,n+1}$$

& hence an area

$$S_n = \pi\, l_B^2\,(2\,n+1)$$

For a particle in state $|\,n\,m\rangle$, its guiding center is on the circle C_m of radius R_m centered at the origin. Its orbit covers a ring straddling the circle C_m. In order to avoid overlaps, the area between successive rings must be at least S_n. The separation $\Delta\,m$ between allowable values of m is therefore given by

$$\pi\left(R_m^2 - R_{m-\Delta\,m}^2\right) \geq S_n$$

$$\pi\, l_B^2\,[\,(2\,m+1) - (2\,m - 2\,\Delta\,m - 1)] \geq \pi\, l_B^2\,(2\,n+1)$$

→ $2\,\Delta\,m \geq 2\,n - 1$

Since $\Delta\,m$ must be a positive integer, we have

$$\Delta\,m = n+1$$

Thus, the area between adjacent rings is

$$\pi\left(R_m^2 - R_{m-n-1}^2\right) = \pi\, l_B^2\,[\,2\,m+1 - (2\,m - 2\,n - 1)\,]$$
$$= 2\,\pi\, l_B^2\,(n+1)$$

so that the flux trapped there is

$$2\,\pi\, l_B^2\,(n+1)\;B = (n+1)\,\Phi_0$$

as befits an n^{th} Landau level.

The area enclosed by the smallest circle must be smaller or equal to $2\,\pi\, l_B^2\,(n+1)$, i.e.,

$$2\,\pi\, l_B^2\left(m_{\min} + \frac{1}{2}\right) \leq 2\,\pi\, l_B^2\,(n+1)$$

→ $m_{\min} \leq n + \frac{1}{2}$

Since $m_{\min}$ must be a positive integer, we have $m_{\min} = n$. The largest possible value was chosen to allow the most room for the cyclotron orbit with center on $C_{m_{\min}}$. Note that with this choice,

$$R_{m_{\min}} = l_B\sqrt{2\,n+1} = R_{c\,n}$$

which seems reasonable geometrically.

∴ $m = n + p\,(n+1) \qquad p = 0,\ 1,\ 2,\ \ldots$

$= n,\ 2\,n+1,\ 3\,n+2,\ \ldots.$

Density of States

$$E = \frac{1}{2} M \boldsymbol{v}^2 = \left(n + \frac{1}{2}\right) \hbar\, \omega_c$$

Setting $\hbar\, \boldsymbol{k} = M\, \boldsymbol{v}$

gives $$\boldsymbol{k}_n^2 = (2\,n+1)\,\frac{M\omega_c}{\hbar} = (2\,n+1)\,\frac{|\,e\,|\;B}{\hbar\,c} = \frac{2\,n+1}{l_B^2}$$

The n^{th} Landau level is therefore just a circle of radius k_n in the 2-D k-space.
This means the originally (when $\boldsymbol{B} = 0$) uniformly distributed states in the k-space are now swept into these discrete circles. To be systematic, we'll let the states with $k_n \geq k > k_{n-1}$ be swept into level n at k_n, with the understanding that $k_{-1} = 0$.

The area between k_n & k_{n-1} is

$$\Delta S_n = \pi\, k_n^2 - \pi\, k_{n-1}^2 = \frac{\pi}{l_B^2}\{\,2\,n+1-(2\,n-1)\,\} = \frac{2\,\pi}{l_B^2}$$

with the exception that

$$\Delta S_0 = \frac{\pi}{l_B^2}$$

In order to get rid of the exception at $n = 0$, we consider the general case of sweeping a fraction α of the states between k_{n+1} & k_n and β of those between k_n & k_{n-1} into level n. Obviously, $\alpha + \beta = 1$ if every state is swept. Applying the prescription down to $n = 1$, we see that level $n = 0$ contains α of the states between k_1 & k_0 and all the states from k_0 to 0. Hence,

$$\Delta S_n = \alpha\,\pi\left(k_{n+1}^2 - k_n^2\right) + \beta\,\pi\left(k_n^2 - k_{n-1}^2\right)$$
$$= \frac{\pi}{l_B^2}\{\,\alpha(2\,n+3-2\,n-1) + \beta(2\,n+1-2\,n+1)\,\} = \frac{2\,\pi}{l_B^2} \qquad \forall\; n \geq 1$$

& $$\Delta S_0 = \alpha\,\pi\left(k_1^2 - k_0^2\right) + \pi\,k_0^2 = \frac{\pi}{l_B^2}\{\,\alpha(3-1)+1\} = \frac{\pi}{l_B^2}(2\,\alpha+1)$$

Thus, $\Delta S_n = \frac{2\,\pi}{l_B^2}$ irregardless of the value of α, while $\frac{\pi}{l_B^2} \leq \Delta S_0 \leq \frac{3\,\pi}{l_B^2}$.

Choosing $\alpha = \frac{1}{2}$, we have

$$\Delta S_n = \frac{2\,\pi}{l_B^2} \qquad \forall\; n \geq 0$$

For a system of dimensions $L \times L$, the original (when $\boldsymbol{B} = 0$) k-space density of states is

$$D(\boldsymbol{k}) = \frac{L^2}{(2\,\pi)^2}$$

Thus, the number of states swept into the n^{th} Landau level is

$$N_n = D(\boldsymbol{k})\,\Delta S_n = \frac{L^2}{2\,\pi\,l_B^2}$$

Since it is independent of n but is B dependent, we write it as

$$N_\Phi = \frac{L^2}{2\,\pi\,l_B^2}$$

$$= \frac{L^2 B}{\Phi_0} = \frac{\Phi}{\Phi_0} = \text{total flux in units of } \Phi_0$$

= Number of Dirac flux quanta in the system
= Number of states in each Landau level
= Maximum number of fermions that can occupy each Landau level

In real space, these states are distributed in an area L^2.
The density (in real space) of Landau states in each Landau level is therefore

$$\rho_\Phi = \frac{N_\Phi}{L^2} = \frac{1}{2\pi l_B^2}$$

i.e., each Landau state (or Dirac flux quantum) occupies an area of $2\pi l_B^2$, as previously shown.

When there are N particles in the system, the filling factor is

$$\nu = \frac{N}{N_\Phi} = \frac{\rho_0}{\rho_\Phi} = 2\pi l_B^2 \rho_0 = \frac{2\pi \hbar c \rho_0}{|e| B}$$

where ρ_0 is the average (or $\boldsymbol{B} = 0$) number density of particles.

Note that the label of the highest occupied Landau level is given by

Floor (ν) = Largest integer that is equal or smaller than ν.

Using

$$N = \frac{L^2}{(2\pi)^2} \pi k_F^2 \rightarrow \qquad k_F = 2\sqrt{\frac{\pi N}{L^2}} = 2\sqrt{\pi \rho_0}$$

we see that for particles at the Fermi level,

$$\nu = 2\pi l_B^2 \rho_0 = \frac{1}{2} k_F^2 l_B^2$$

so that the cyclotron radius is roughly

$$R_c \simeq l_B \sqrt{2\nu + 1} \simeq k_F l_B^2 = \frac{\hbar c}{|e| B} k_F$$

Thus, the cyclotron covers roughly an area of

$$\pi R_c^2 \simeq 2\pi l_B^2 \left(\frac{1}{\sqrt{2}} l_B k_F \right)^2$$

& $\left(\frac{1}{\sqrt{2}} l_B k_F \right)^2$ flux quanta.

Using

$$N_\Phi = \frac{L^2}{2\pi l_B^2} = \frac{2N}{l_B^2 k_F^2}$$

we have

$$\left(\frac{1}{\sqrt{2}} l_B k_F \right)^2 = \frac{N}{N_\Phi} = \nu$$

In summary, a particle at the fermi level occupies a state at Landau level Floor(ν), while its cyclotron orbit in real space covers roughly ν flux quanta.