10.3. Symmetric Gauge

Symmetric gauge:

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r} = \frac{1}{2}B\,\hat{\boldsymbol{z}}\times\boldsymbol{r} = \frac{1}{2}B\,(-y,\,x,\,0)$$

i.e.,
$$A_i = \frac{1}{2}\varepsilon_{ijk}B_j\,r_k = -\frac{1}{2}B\,\varepsilon_{ij3}\,r_j$$

∴
$$\boldsymbol{D} = \nabla - i\frac{eB}{2\hbar c}\hat{\boldsymbol{z}}\times\boldsymbol{r} = \nabla - i\frac{\mathbb{e}}{2\,l_B^2}\hat{\boldsymbol{z}}\times\boldsymbol{r} \qquad \left(l_B^2 = \frac{\hbar c}{|\,e\,|\,B}\right)$$

$$= \left(\partial_x + i\frac{\mathbb{e}}{2\,l_B^2}y\,,\ \partial_y - i\frac{\mathbb{e}}{2\,l_B^2}x\right)$$

$$\boldsymbol{v} = \frac{\hbar}{M\,i}\nabla - \frac{eB}{2Mc}\hat{\boldsymbol{z}}\times\boldsymbol{r} = \frac{\hbar}{M\,i}\nabla - \frac{\mathbb{e}}{2}\omega_c\,\hat{\boldsymbol{z}}\times\boldsymbol{r} \qquad \left(\omega_c = \frac{|\,e\,|\,B}{M\,c} = \frac{\hbar}{M\,l_B^2}\right)$$

$$= \left(\frac{\hbar}{M\,i}\partial_x + \frac{\mathbb{e}}{2}\omega_c\,y\,,\ \frac{\hbar}{M\,i}\partial_y - \frac{\mathbb{e}}{2}\omega_c\,x\right)$$

$$v_i = \frac{\hbar}{M\,i}\partial_i + \frac{\mathbb{e}}{2}\omega_c\,\varepsilon_{ij3}\,r_j$$

→
$$\boldsymbol{v}^2 = -\frac{\hbar^2}{M^2}\left\{\left(\partial_x + i\frac{\mathbb{e}}{2\,l_B^2}y\right)^2 + \left(\partial_y - i\frac{\mathbb{e}}{2\,l_B^2}x\right)^2\right\}$$

$$= -\frac{\hbar^2}{M^2}\left\{\partial_x^2 + \partial_y^2 - \frac{1}{4\,l_B^4}(x^2+y^2) + i\frac{\mathbb{e}}{l_B^2}(y\,\partial_x - x\,\partial_y)\right\}$$

∴
$$\frac{1}{\omega_c^2}\boldsymbol{v}^2 = -l_B^4\left\{\partial_x^2 + \partial_y^2 - \frac{1}{4\,l_B^4}(x^2+y^2) + i\frac{\mathbb{e}}{l_B^2}(y\,\partial_x - x\,\partial_y)\right\}$$

$$= \frac{1}{4}(x^2+y^2) - l_B^4\left(\partial_x^2+\partial_y^2\right) + i\,\mathbb{e}\,l_B^2(x\,\partial_y - y\,\partial_x)$$

$$\boldsymbol{R} = \boldsymbol{r} - \frac{\mathbb{e}}{\omega_c}\hat{\boldsymbol{z}}\times\boldsymbol{v} = \boldsymbol{r} - \frac{\mathbb{e}}{\omega_c}\hat{\boldsymbol{z}}\times\left(\frac{\hbar}{M\,i}\nabla - \frac{\mathbb{e}}{2}\omega_c\,\hat{\boldsymbol{z}}\times\boldsymbol{r}\right)$$

$$= \boldsymbol{r} + \hat{\boldsymbol{z}}\times\left(i\,\mathbb{e}\,l_B^2\nabla + \frac{1}{2}\hat{\boldsymbol{z}}\times\boldsymbol{r}\right)$$

$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$ → $\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r}) = -\boldsymbol{r}$ for $\boldsymbol{r}$ in xy-plane

∴
$$\boldsymbol{R} = \frac{1}{2}\boldsymbol{r} + i\,\mathbb{e}\,l_B^2\,\hat{\boldsymbol{z}}\times\nabla = \left(\frac{1}{2}x - i\,\mathbb{e}\,l_B^2\,\partial_y\,,\ \frac{1}{2}y + i\,\mathbb{e}\,l_B^2\,\partial_x\right)$$

$$R_i = \frac{1}{2}r_i - i\,\mathbb{e}\,l_B^2\,\varepsilon_{ij3}\,\partial_j$$

→
$$\boldsymbol{R}^2 = \left(\frac{1}{2}x - i\,\mathbb{e}\,l_B^2\,\partial_y\right)^2 + \left(\frac{1}{2}y + i\,\mathbb{e}\,l_B^2\,\partial_x\right)^2$$

$$= \frac{1}{4}(x^2+y^2) - l_B^4\left(\partial_x^2+\partial_y^2\right) - i\,\mathbb{e}\,l_B^2\,(x\,\partial_y - y\,\partial_x)$$

$$= \frac{1}{\omega_c^2}\boldsymbol{v}^2 - 2\,i\,\mathbb{e}\,l_B^2\,(x\,\partial_y - y\,\partial_x)$$

$$= \frac{1}{\omega_c^2} \boldsymbol{v}^2 + 2\,\mathbb{e}\, l_B^2 \frac{L_3}{\hbar}$$

$$\therefore \qquad L_3 = \mathbb{e} \frac{\hbar}{2\, l_B^2} \left(\boldsymbol{R}^2 - \frac{1}{\omega_c^2} \boldsymbol{v}^2 \right)$$

$$= \frac{1}{2} \left(\frac{e\,B}{c} \boldsymbol{R}^2 - \frac{c\,M^2}{e\,B} \boldsymbol{v}^2 \right)$$

$$= \mathbb{e} \left(\frac{1}{2} M\, \omega_c \boldsymbol{R}^2 - \frac{1}{\omega_c} H \right)$$

Using (see 10.3.a._AngularMomentum.pdf),

$$H = -\mathbb{e} \frac{1}{2} \omega_c \mathcal{L}_3$$

we have

$$L_3 = \frac{1}{2} \left(\mathbb{e}\, M\, \omega_c \boldsymbol{R}^2 + \mathcal{L}_3 \right)$$

On the other hand, classical parallel axis theorem gives

$$L_3 = -\mathbb{e}\, M\, \omega_c \boldsymbol{R}^2 + \mathcal{L}_3$$

The failure of this theorem is due to the fact that $\left[R_i, R_j\right] \neq 0$ implies that if $R = |\boldsymbol{R}|$ is fixed precisely, then the angular position of $\boldsymbol{R}$ is completely lost, i.e., the guiding center can be anywhere on the circle of radius R centered at the origin.

Also,

$$\boldsymbol{R}^2 = 2\, l_B^2 \left(b^+ b + \frac{1}{2} \right) \qquad\qquad H = \frac{1}{2} M\, \boldsymbol{v}^2 = \hbar\, \omega_c \left(a^+ a + \frac{1}{2} \right)$$

$$\to \qquad L_3 = \mathbb{e}\, \hbar \left(b^+ b - a^+ a \right)$$

The stationary states of the system can be written as $|\, n, m \rangle$ with

$$a^+ a\, |\, n, m \rangle = N\, |\, n, m \rangle \qquad \& \qquad b^+ b\, |\, n, m \rangle = m\, |\, n, m \rangle$$

where n is the label of the Landau level.

$$\to \qquad L_3\, |\, n, m \rangle = \mathbb{e}\, \hbar\, (m - n)\, |\, n, m \rangle$$

where (see 10.2._CyclotronMotion.pdf)

$$n = 0, 1, 2, \ldots \qquad m = n + p(n+1) \qquad p = 0, 1, 2, \ldots$$

i.e., in the n^{th} Landau level,

$$\frac{|\, L_3\, |}{\hbar} = 0,\; n+1,\; 2\,(n+1),\; \ldots.$$

Thus, in the lowest Landau level,

$$L_3\, |\, 0, m \rangle = \mathbb{e}\, \hbar\, m\, |\, 0, m \rangle$$

so that m denotes the angular momentum.

Note that since both $\boldsymbol{R}^2$ & H are bounded below,

$$\langle n', m' \,|\, n, m \rangle = \delta_{n' n}\, \delta_{m' m}$$

$$\to \qquad |\, n, m \rangle = \frac{a^{+n}}{\sqrt{n!}} \frac{b^{+m}}{\sqrt{m!}} \left| \, 0 \right\rangle$$

Motion in the Complex Plane

$$H = \frac{1}{2} M \mathbf{v}^2 = \hbar\,\omega_c \left(a^+ a + \frac{1}{2}\right)$$

$$a = \sqrt{\frac{M}{2\,\hbar\,\omega_c}}\ (v_1 + i\,\mathbb{e}\,v_2) = \frac{1}{\sqrt{2}\ \omega_c\,l_B}\ (v_1 + i\,\mathbb{e}\,v_2)$$

$$a^+ = \sqrt{\frac{M}{2\,\hbar\,\omega_c}}\ (v_1 - i\,\mathbb{e}\,v_2) = \frac{1}{\sqrt{2}\ \omega_c\,l_B}\ (v_1 - i\,\mathbb{e}\,v_2)$$

Let

$$z = x + i\,y \qquad\qquad z^* = x - i\,y$$

Taken as an operator, z is not hermitian & hence not observable. However, it behaves like the position operator of the particle if we treat the xy-plane as the complex plane.

$$\rightarrow \qquad x = \frac{1}{2}(z + z^*) \qquad\qquad y = \frac{1}{2\,i}(z - z^*)$$

$$\therefore \qquad \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Treating z as an operator,

$$[z, z^+] = 0$$

$$\left[z, \frac{\partial}{\partial z}\right] = z\frac{\partial}{\partial z} - \frac{\partial}{\partial z} z = -1 \qquad\qquad \left[z^*, \frac{\partial}{\partial z^*}\right] = -1$$

$$\left[z, \frac{\partial}{\partial z^*}\right] = z\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z^*} z = 0 = \left[z^*, \frac{\partial}{\partial z}\right]$$

In the symmetric gauge,

$$\mathbf{v} = \left(\frac{\hbar}{M\,i}\partial_x + \frac{\mathbb{e}}{2}\,\omega_c\,y\,,\ \frac{\hbar}{M\,i}\partial_y - \frac{\mathbb{e}}{2}\,\omega_c\,x\right)$$

$$\rightarrow \qquad a = \sqrt{\frac{M}{2\,\hbar\,\omega_c}}\ \left\{\frac{\hbar}{M\,i}\partial_x + \frac{\mathbb{e}}{2}\,\omega_c\,y + i\,\mathbb{e}\left(\frac{\hbar}{M\,i}\partial_y - \frac{\mathbb{e}}{2}\,\omega_c\,x\right)\right\}$$

$$= -i\sqrt{\frac{\hbar}{2\,M\,\omega_c}}\ \left(\partial_x + i\,\mathbb{e}\,\partial_y\right) - i\sqrt{\frac{M\,\omega_c}{2\,\hbar}}\ \frac{1}{2}(x + i\,\mathbb{e}\,y)$$

$$= -\frac{i}{\sqrt{2}}\left(\frac{1}{2\,l_B}(x + i\,\mathbb{e}\,y) + l_B\left(\partial_x + i\,\mathbb{e}\,\partial_y\right)\right)$$

$$a^+ = \frac{i}{\sqrt{2}}\left(\frac{1}{2\,l_B}(x - i\,\mathbb{e}\,y) - l_B\left(\partial_x - i\,\mathbb{e}\,\partial_y\right)\right)$$

In order to handle the sign $\mathbb{e}$, we set

$$\mathbb{z} = \begin{cases} z & \text{if } \mathbb{e} = 1 \\ z^* & \text{if } \mathbb{e} = -1 \end{cases} \qquad\qquad \mathbb{z}^* = \begin{cases} z^* & \text{if } \mathbb{e} = 1 \\ z & \text{if } \mathbb{e} = -1 \end{cases}$$

$$\frac{\partial}{\partial \mathbb{z}} = \begin{cases} \frac{\partial}{\partial z} & \text{if } \mathbb{e} = 1 \\ \frac{\partial}{\partial z^*} & \text{if } \mathbb{e} = -1 \end{cases} \qquad\qquad \frac{\partial}{\partial \mathbb{z}^*} = \begin{cases} \frac{\partial}{\partial z^*} & \text{if } \mathbb{e} = 1 \\ \frac{\partial}{\partial z} & \text{if } \mathbb{e} = -1 \end{cases}$$

$$\rightarrow \qquad a = -\frac{i}{2\sqrt{2}}\sqrt{\frac{M\,\omega_c}{\hbar}}\left(\mathbb{z} + \frac{4\,\hbar}{M\,\omega_c}\frac{\partial}{\partial \mathbb{z}^*}\right)$$

$$= -\frac{i}{2\sqrt{2}\, l_B}\left(z + 4 l_B^2 \frac{\partial}{\partial z^*}\right)$$
$$= -\frac{i}{2\sqrt{2}}\left(\tilde{z} + 4 \frac{\partial}{\partial \tilde{z}^*}\right)$$

where $\tilde{z} = \frac{1}{l_B} z$ $\qquad$ $\tilde{z}^* = \frac{1}{l_B} z$

→ $\frac{\partial}{\partial \tilde{z}} = l_B \frac{\partial}{\partial z}$ $\qquad$ $\left[\frac{\partial}{\partial \tilde{z}}, \tilde{z}\right] = 1$

∴ $a^+ = \frac{i}{2\sqrt{2}}\left(\tilde{z}^* - 4 \frac{\partial}{\partial \tilde{z}}\right)$ $\qquad$ $\left(\left(\frac{\partial}{\partial \tilde{z}^*}\right)^+ = \frac{\overleftarrow{\partial}}{\partial \tilde{z}} = -\frac{\partial}{\partial \tilde{z}}\right)$

$$a a^+ = \frac{1}{8}\left(\tilde{z}\tilde{z}^* - 16 \frac{\partial}{\partial \tilde{z}^*}\frac{\partial}{\partial \tilde{z}}\right) + \frac{1}{2}\left(\frac{\partial}{\partial \tilde{z}^*}\tilde{z}^* - \tilde{z}\frac{\partial}{\partial \tilde{z}}\right)$$
$$= \frac{1}{8}\left(\tilde{z}\tilde{z}^* - 16 \frac{\partial}{\partial \tilde{z}^*}\frac{\partial}{\partial \tilde{z}}\right) + \frac{1}{2}\left(1 + \tilde{z}^*\frac{\partial}{\partial \tilde{z}^*} - \tilde{z}\frac{\partial}{\partial \tilde{z}}\right)$$
$$a^+ a = \frac{1}{8}\left(\tilde{z}\tilde{z}^* - 16 \frac{\partial}{\partial \tilde{z}^*}\frac{\partial}{\partial \tilde{z}}\right) + \frac{1}{2}\left(-1 + \tilde{z}^*\frac{\partial}{\partial \tilde{z}^*} - \tilde{z}\frac{\partial}{\partial \tilde{z}}\right)$$

→ $[a, a^+] = 1$

$$a a^+ + a^+ a = \frac{1}{4}\left(\tilde{z}\tilde{z}^* - 16 \frac{\partial}{\partial \tilde{z}^*}\frac{\partial}{\partial \tilde{z}}\right) + \tilde{z}^*\frac{\partial}{\partial \tilde{z}^*} - \tilde{z}\frac{\partial}{\partial \tilde{z}}$$

Consider

$$a^+ a \mid n\rangle = n \mid n\rangle$$

The ground state $\mid 0\rangle$ is given by

$$a \mid 0\rangle = 0$$

→ $-\frac{i}{2\sqrt{2}}\left(\tilde{z} + 4 \frac{\partial}{\partial \tilde{z}^*}\right)\langle \tilde{z} \mid 0\rangle = 0$

i.e. $\left(\tilde{z} + 4 \frac{\partial}{\partial \tilde{z}^*}\right)\phi_0(\tilde{z}) = 0$

→ $\phi_0(z) = \lambda_0(\tilde{z}) \exp\left(-\frac{1}{4}\tilde{z}^*\tilde{z}\right) = \lambda_0 \exp\left(-\frac{1}{4}\tilde{r}^2\right)$ $\qquad$ $\left(\tilde{z} = \tilde{r}\, e^{i\,\theta} = \frac{r}{l_B} e^{i\,\theta}\right)$

where $\lambda_0(\tilde{z})$ is an analytic function of $\tilde{z}$ so that $\frac{\partial \lambda_0}{\partial \tilde{z}^*} = 0$.

(This general form will be used in $\mid n\, m\rangle$ discussed below.)

The inner product is defined as

$$\langle \phi \mid \psi\rangle = \int dx \int dy\, \phi^* \psi = \int r\, dr \int d\theta\, \phi^* \psi = l_B^2 \int \tilde{r}\, d\tilde{r} \int d\theta\, \phi^* \psi$$

$\tilde{z} = \tilde{r}\, e^{i\,\theta}$ $\qquad$ → $\qquad$ $\tilde{z}^* = \tilde{r}\, e^{-i\,\theta}$

∴ $\frac{\partial(\tilde{z}^*, \tilde{z})}{\partial(\tilde{r}, \theta)} = \begin{vmatrix} \partial_{\tilde{r}}\tilde{z}^* & \partial_\theta \tilde{z}^* \\ \partial_{\tilde{r}}\tilde{z} & \partial_\theta \tilde{z} \end{vmatrix} = \begin{vmatrix} e^{-i\,\theta} & -i\,\tilde{r}\, e^{-i\,\theta} \\ e^{i\,\theta} & i\,\tilde{r}\, e^{i\,\theta} \end{vmatrix} = 2\, i\, \tilde{r}$

$$\int d\tilde{z}^* \int d\tilde{z} = 2\, i \int_0^\infty \tilde{r}\, d\tilde{r} \int_0^{2\pi} d\theta = \frac{2\, i}{l_B^2}\int_0^\infty r\, dr \int_0^{2\pi} d\theta$$

→ $\langle \phi \mid \psi\rangle = \frac{l_B^2}{2\, i}\int d\tilde{z}^* \int d\tilde{z}\, \phi^* \psi$

For simplicity, we set $\lambda_0(\mathbb{z}) = C$.

→ $$\langle 0 \mid 0 \rangle = C^* C\, 2\pi\, l_B^2 \int_0^\infty \tilde{r}\, d\tilde{r}\, \exp\left(-\frac{1}{2} \tilde{r}^2\right) = C^* C\, 2\pi\, l_B^2 = 1$$

∴ $$\phi_0(\mathbb{z}) = \frac{1}{\sqrt{2\pi\, l_B^2}} \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right) = \frac{1}{\sqrt{2\pi\, l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2\right)$$

Treating the complex plane as the xy-plane, we have

$$\phi_0(\boldsymbol{r}) = \phi_0(\mathbb{z}) = \frac{1}{\sqrt{2\pi\, l_B^2}} \exp\left(-\frac{1}{4\, l_B^2} r^2\right) \qquad \boldsymbol{r} = r\,(\cos\theta,\ \mathbb{e} \sin\theta)$$

$$\mid n \rangle = \frac{a^{+n}}{\sqrt{n!}} \left| 0 \right\rangle$$

$$\phi_n(\mathbb{z}) = \frac{a^{+n}}{\sqrt{n!}} \langle \mathbb{z} \mid 0 \rangle$$

$$= \frac{1}{\sqrt{2\pi\, l_B^2\, n!}} \left(\frac{i}{2\sqrt{2}}\right)^n \left(\tilde{\mathbb{z}}^* - 4 \frac{\partial}{\partial \tilde{\mathbb{z}}}\right)^n \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right)$$

$$\frac{\partial}{\partial \tilde{\mathbb{z}}} \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right) = -\frac{1}{4} \tilde{\mathbb{z}}^* \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right)$$

$$\left[\tilde{\mathbb{z}}^*,\ \frac{\partial}{\partial \tilde{\mathbb{z}}}\right] = 0$$

→ $$\left(\tilde{\mathbb{z}}^* - 4 \frac{\partial}{\partial \tilde{\mathbb{z}}}\right)^n \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right) = (2\, \tilde{\mathbb{z}}^*)^n \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right)$$

∴ $$\phi_n(\mathbb{z}) = \frac{1}{\sqrt{n!\, 2\pi\, l_B^2}} \left(\frac{i}{\sqrt{2}} \tilde{\mathbb{z}}^*\right)^n \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^*\right)$$

$$\phi_n(\boldsymbol{r}) = \frac{1}{\sqrt{n!\, 2\pi\, l_B^2}} \left(\frac{i}{\sqrt{2}} \tilde{r}\right)^n e^{-i\, \mathbb{e}\, n\, \theta} \exp\left(-\frac{1}{4} \tilde{r}^2\right)$$

$$= \frac{1}{\sqrt{n!\, 2\pi\, l_B^2}} \left(\frac{i\, r}{\sqrt{2}\, l_B}\right)^n e^{-i\, \mathbb{e}\, n\, \theta} \exp\left(-\frac{1}{4\, l_B^2} r^2\right)$$

→ $$\langle n \mid n' \rangle = \int_0^{2\pi} d\theta \int_0^\infty d r\, r\, \phi_n^*(\boldsymbol{r})\, \phi_{n'}(\boldsymbol{r})$$

$$= l_B^2 \int_0^{2\pi} d\theta \int_0^\infty d\tilde{r}\, \tilde{r}\, \phi_n^*(\boldsymbol{r})\, \phi_{n'}(\boldsymbol{r})$$

$$= \frac{i^{n-n'}}{2\pi \sqrt{2^{n+n'}\, n!\, n'!}} \int_0^{2\pi} d\theta\, e^{i\, \mathbb{e} (n-n')\, \theta} \int_0^\infty d\tilde{r}\, \tilde{r}^{n+n'+1} \exp\left(-\frac{1}{2} \tilde{r}^2\right)$$

$$= \delta_{n n'} \frac{1}{2^n\, n!} \int_0^\infty d\tilde{r}\, \tilde{r}^{2n+1} \exp\left(-\frac{1}{2} \tilde{r}^2\right)$$

$$= \delta_{n n'}$$

Similarly, with

$$b = \frac{1}{\sqrt{2}\, l_B} (R_1 - i\, \mathbb{e}\, R_2)$$

$$= \frac{1}{\sqrt{2}\, l_B} \left\{ \frac{1}{2} x - i\, \mathbb{e}\, l_B^2 \partial_y - i\, \mathbb{e} \left(\frac{1}{2} y + i\, \mathbb{e}\, l_B^2 \partial_x \right) \right\}$$

$$= \frac{1}{2\sqrt{2}\, l_B} \left(\mathbb{z}^* + 4\, l_B^2 \frac{\partial}{\partial \mathbb{z}} \right)$$

$$= \frac{1}{2\sqrt{2}} \left(\tilde{\mathbb{z}}^* + 4 \frac{\partial}{\partial \tilde{\mathbb{z}}} \right)$$

→ $$b^+ = \frac{1}{2\sqrt{2}} \left(\tilde{\mathbb{z}} - 4 \frac{\partial}{\partial \tilde{\mathbb{z}}^*} \right)$$

Thus

$i\, a \leftrightarrow b$ if $\quad \mathbb{z} \leftrightarrow \mathbb{z}^*$ or $\quad \mathbb{e} \leftrightarrow -\mathbb{e}$

$b \mid 0 \rangle = 0$

→ $$\frac{1}{2\sqrt{2}} \left(\tilde{\mathbb{z}}^* + 4 \frac{\partial}{\partial \tilde{\mathbb{z}}} \right) \langle \mathbb{z} \mid 0 \rangle = 0$$

i.e. $$\left(\tilde{\mathbb{z}}^* + 4 \frac{\partial}{\partial \tilde{\mathbb{z}}} \right) \psi_0(\mathbb{z}) = 0$$

→ $$\psi_0(\mathbb{z}) = \phi_0(\mathbb{z}) = \frac{1}{\sqrt{2\pi\, l_B^2}} \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^* \right) = \langle \boldsymbol{r} \mid 0, 0 \rangle$$

Treating the complex plane as the xy-plane, we have

$$\psi_0(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi\, l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2 \right) \qquad \boldsymbol{r} = r\, e^{i\, \mathbb{e}\, \theta} = r\, (\cos\theta,\ \mathbb{e} \sin\theta)$$

$$\mid m \rangle = \frac{b^{+m}}{\sqrt{m!}} \left| 0 \right\rangle$$

$$\psi_m(\mathbb{z}) = \frac{b^{+m}}{\sqrt{m!}} \langle \mathbb{z} \mid 0 \rangle$$

$$= \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{1}{2\sqrt{2}} \right)^m \left(\tilde{\mathbb{z}} - 4 \frac{\partial}{\partial \tilde{\mathbb{z}}^*} \right)^m \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^* \right)$$

$$= \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{1}{\sqrt{2}} \tilde{\mathbb{z}} \right)^m \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^* \right)$$

$$\psi_m(\boldsymbol{r}) = \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{1}{\sqrt{2}} \tilde{r} \right)^m e^{i\, \mathbb{e}\, m\, \theta} \exp\left(-\frac{1}{4} \tilde{r}^2 \right)$$

with $$\langle m \mid m \rangle = \int_0^{2\pi} d\theta \int_0^\infty d r\, r\, \psi_m^*(\boldsymbol{r})\, \psi_m(\boldsymbol{r}) = 1$$

→ $$\langle m \mid m' \rangle = \int_0^{2\pi} d\theta \int_0^\infty d r\, r\, \psi_m^*(\boldsymbol{r})\, \psi_{m'}(\boldsymbol{r})$$

$$= l_B^2 \int_0^{2\pi} d\theta \int_0^\infty d\tilde{r}\, \tilde{r}\, \psi_m^*(\boldsymbol{r})\, \psi_m(\boldsymbol{r})$$

$$= \frac{1}{2\pi \sqrt{2^{m+m'}\, m!\, m'!}} \int_0^{2\pi} d\theta\, e^{i\, \mathbb{e} (m'-m)\, \theta} \int_0^\infty d\tilde{r}\, \tilde{r}^{m+m'+1} \exp\left(-\frac{1}{2} \tilde{r}^2 \right)$$

$$= \delta_{m m'} \frac{1}{2^m\, m!} \int_0^\infty d\,\tilde r\; \tilde r^{\,2m+1} \exp\left(-\frac{1}{2}\tilde r^2\right)$$
$$= \delta_{m m'}$$

$$\rightarrow \qquad \Psi_{n m}(\boldsymbol r) = \frac{1}{\sqrt{(n+m)!\, 2\pi\, l_B^2}} \left(\frac{i}{\sqrt 2}\tilde{\mathbb z}^*\right)^n \left(\frac{\tilde{\mathbb z}}{\sqrt 2}\right)^m \exp\left(-\frac{1}{4}\tilde{\mathbb z}\,\tilde{\mathbb z}^*\right)$$
$$= \frac{i^n}{\sqrt{(n+m)!\, 2\pi\, l_B^2}} \left(\frac{\tilde r}{\sqrt 2}\right)^{n+m} e^{i\,\mathbb e\,(m-n)\,\theta} \exp\left(-\frac{1}{4}\tilde r^2\right)$$

$$\therefore \qquad \langle n\, m \mid n'\, m' \rangle = \int_0^{2\pi} d\,\theta \int_0^\infty d\,r\, r\, \psi^*_{n m}(\boldsymbol r)\, \psi_{n' m'}(\boldsymbol r)$$
$$= l_B^2 \int_0^{2\pi} d\,\theta \int_0^\infty d\,\tilde r\; \tilde r\; \psi^*_{n m}(\boldsymbol r)\, \psi_{n' m'}(\boldsymbol r)$$

$$= \frac{i^{\,n-n'}}{\pi\sqrt{2^{n+m}\,(n+m)!\,(n'+m')!}} \int_0^{2\pi} d\,\theta\, e^{i\,\mathbb e (n-n'+m'-m)\,\theta} \int_0^\infty d\,\tilde r\; \tilde r^{\,n+n'+m+m'+1} \exp\left(-\frac{1}{2}\tilde r^2\right)$$
$$= \delta_{n+m',\, n'+m} \frac{2}{\pi\sqrt{2^{n+m}\,(n+m)!\,(n'+m')!}} \int_0^\infty d\,\tilde r\; \tilde r^{\,2(n+m')+1} \exp\left(-\frac{1}{2}\tilde r^2\right)$$
$$= \delta_{n+m',\, n'+m} \frac{(n+m')!}{\sqrt{(n+m)!\,(n'+m')!}}$$

$$\rightarrow \qquad \langle n\, m \mid n'\, m' \rangle = \begin{cases} 1 & \text{if } n = n' \;\; \& \;\; m = m' \\ (n+m')!/(\sqrt{((n+m)!\,(n'+m')!)}) & \text{if } n - m = n' - m' \end{cases}$$

Since $L_3 = \mathbb e\,(m-n)\,\hbar$, all states with the same L_3 are non-orthogonal.

The probability of finding the particle at $\boldsymbol r$ is

$$\rho_{n m}(\boldsymbol r) = \Psi^*_{n m}(\boldsymbol r)\, \Psi_{n m}(\boldsymbol r)$$
$$\propto \left(\frac{\tilde r}{\sqrt 2}\right)^{2(n+m)} \exp\left(-\frac{1}{2}\tilde r^2\right)$$

$$\rho_{n m}' = \left(\frac{2(n+m)}{\tilde r} - \tilde r\right) \left(\frac{\tilde r}{\sqrt 2}\right)^{2(n+m)} \exp\left(-\frac{1}{2}\tilde r^2\right) = 0$$

$\rightarrow$ $\rho_{n m}$ is peaked at

$$\tilde r_{n m} = \sqrt{2(n+m)} \qquad \text{i.e.,} \qquad r_{n m} = \sqrt{2(n+m)}\; l_B.$$

The Noether current density is (see 5.3._InteractionWithMatterField.pdf)

$$\boldsymbol J = i\frac{e\hbar}{2M}[\,(\boldsymbol D\psi)^+\,\psi - \psi^+\,\boldsymbol D\psi\,] \qquad \text{(only space part considered \& set } f = \frac{\hbar^2}{2M}\text{)}$$

Using (see 8.3._QuantumMechanics.pdf)

$$\nabla\theta(\boldsymbol r) = -\frac{1}{r^2}\boldsymbol r\times\hat z = \frac{1}{r}\hat{\boldsymbol\theta}(\boldsymbol r) \qquad\qquad \nabla r = \hat{\boldsymbol r}$$

we see that

$$\boldsymbol D = \nabla - i\frac{\mathbb e}{2 l_B^2}\hat{\boldsymbol z}\times\boldsymbol r = \nabla - i\frac{\mathbb e\, r}{2 l_B^2}\hat{\boldsymbol\theta}$$
$$= \frac{1}{l_B}\left(\tilde\nabla - \frac{1}{2} i\,\mathbb e\,\tilde r\,\hat{\boldsymbol\theta}\right) \qquad \tilde\nabla = \hat{\boldsymbol r}\,\partial_{\tilde r} + \hat{\boldsymbol\theta}\frac{1}{\tilde r}\partial_\theta \qquad \tilde\nabla\tilde r = \hat{\boldsymbol r}$$

For $\psi = \psi_{n\,m}$, we have

$$\boldsymbol{D}\,\psi_{n\,m} = \frac{1}{l_B}\left(\frac{n+m}{\tilde{r}}\,\hat{\boldsymbol{r}} - \frac{1}{2}\,\tilde{r}\,\hat{\boldsymbol{r}} + i\,e\,\frac{m-n}{\tilde{r}}\,\hat{\boldsymbol{\theta}} - \frac{1}{2}\,i\,e\,\tilde{r}\,\hat{\boldsymbol{\theta}}\right)\psi_{n\,m}$$

$$(\boldsymbol{D}\,\psi_{n\,m})^{+} = \frac{1}{l_B}\left(\frac{n+m}{\tilde{r}}\,\hat{\boldsymbol{r}} - \frac{1}{2}\,\tilde{r}\,\hat{\boldsymbol{r}} - i\,e\,\frac{m-n}{\tilde{r}}\,\hat{\boldsymbol{\theta}} + \frac{1}{2}\,i\,e\,\tilde{r}\,\hat{\boldsymbol{\theta}}\right)\psi^{*}_{n\,m}$$

$$\to \qquad \boldsymbol{J} = \frac{|\,e\,|\;\hbar}{M\,l_B}\left(\frac{m-n}{\tilde{r}} - \frac{1}{2}\,\tilde{r}\right)\hat{\boldsymbol{\theta}}\,\psi^{*}_{n\,m}\,\psi_{n\,m}$$

$$= \hat{\boldsymbol{\theta}}\,\frac{|\,e\,|\;\hbar}{M\,l_B}\left(\frac{m-n}{\tilde{r}} - \frac{1}{2}\,\tilde{r}\right)\frac{1}{(n+m)!\,2\,\pi\,l_B^2}\left(\frac{\tilde{r}}{\sqrt{2}}\right)^{2(n+m)}\exp\left(-\frac{1}{2}\,\tilde{r}^2\right)$$

$$= \hat{\boldsymbol{\theta}}\,\frac{|\,e\,|\;\hbar}{M}\,\frac{1}{(n+m)!\,\sqrt{2}\;\pi\,l_B^3}\left(m-n-\frac{1}{2}\,\tilde{r}^2\right)\left(\frac{\tilde{r}}{\sqrt{2}}\right)^{2(n+m)-1} e^{-\tilde{r}^2/2}$$

$$= \hat{\boldsymbol{\theta}}\,J_\theta$$

Note: $m = n_{\text{Ezawa}}$ $n = N_{\text{Ezawa}}$ $r = 2\,l_B\,|\,z\,|_{\text{Ezawa}}$

But Ezawa’s eq.10.3.22 can be obtained from $\boldsymbol{J}$ by setting $m = 0$ & $n = n_{\text{Ezawa}}$.

$$\langle \boldsymbol{J} \rangle = \left\langle n,\ m \,\middle|\, \hat{\boldsymbol{\theta}}\,J_\theta \,\middle|\, n,\, m \right\rangle$$

$$= \hat{\boldsymbol{\theta}}\int_0^{2\pi} d\,\theta'\,\delta(\theta'-\theta)\,l_B^2 \int_0^{\infty} d\,\tilde{r}\;\tilde{r}\;J_\theta$$

$$= \hat{\boldsymbol{\theta}}\,\frac{|\,e\,|\;\hbar}{M}\,\frac{1}{(n+m)!\,\pi\;l_B}\int_0^{\infty} d\,\tilde{r}\left(m-n-\frac{1}{2}\,\tilde{r}^2\right)\left(\frac{\tilde{r}}{\sqrt{2}}\right)^{2(n+m)} e^{-\tilde{r}^2/2}$$

$$= -\hat{\boldsymbol{\theta}}\,\frac{|\,e\,|\;\hbar}{M}\,\frac{1}{(n+m)!\,\sqrt{2}\;\pi\,l_B}\left(2\,n+\frac{1}{2}\right)\Gamma\left(m+n+\frac{1}{2}\right)$$

$$= -\hat{\boldsymbol{\theta}}\,\omega_c\,l_B\,\frac{\sqrt{2}}{(n+m)!\,\pi}\left(n+\frac{1}{4}\right)\Gamma\left(m+n+\frac{1}{2}\right)$$

$$\nabla\cdot\boldsymbol{J} = \nabla\cdot\left(J_\theta\,\hat{\boldsymbol{\theta}}\right) = \frac{1}{r}\,\frac{\partial\,J_\theta}{\partial\,\theta} = 0$$

i.e., $\boldsymbol{J}$ is solenoidal.

$$\nabla\times\boldsymbol{J} = \hat{\boldsymbol{z}}\left(\frac{\partial\,J_\theta}{\partial\,r} + \frac{J_\theta}{r}\right)$$

$$= -\hat{\boldsymbol{z}}\,\frac{|\,e\,|\;\hbar}{M}\,\frac{\sqrt{2}}{(n+m)!\,l_B^4}\left[\,\tilde{r}^4 - 2\,(2\,m+1)\,\tilde{r}^2 + 4\left(m^2-n^2\right)\right]\left(\frac{\tilde{r}}{\sqrt{2}}\right)^{2(m+n-1)} e^{-\tilde{r}^2/2}$$

$$\to \qquad \langle\,\nabla\times\boldsymbol{J}\,\rangle = 0$$

Note: The Gaussian integrals can be evaluated using the standard formulae. However, such tedious & error-prone processes are best avoided by using a computer software such as *Mathematica.*