10.3.a. Angular Momentum

From 10.2._CyclotronMotion.pdf ,

$$a = \sqrt{\frac{M}{2\hbar\omega_c}}\,(v_1 + i\,e\,v_2) \qquad a^+ = \sqrt{\frac{M}{2\hbar\omega_c}}\,(v_1 - i\,e\,v_2)$$

$$[a, a^+] = 1 \qquad a\,a^+ + a^+\,a = \frac{M}{\hbar\omega_c}\boldsymbol{v}^2$$

$$\boldsymbol{R} = \boldsymbol{r} - \frac{e}{\omega_c}\hat{\boldsymbol{z}}\times\boldsymbol{v} \qquad R_i = r_i + \frac{e}{\omega_c}\varepsilon_{ij3}\,v_j$$

$$\boldsymbol{r}^+ = \boldsymbol{r} \qquad \boldsymbol{v}^+ = \boldsymbol{v}$$

→ $\boldsymbol{R}^+ = \boldsymbol{R}$

In order to find b such that

$$[b, b^+] = 1 \qquad b\,b^+ + b^+\,b \propto \boldsymbol{R}^2$$

we set

$$b = \alpha(R_1 - i\,e\,R_2) \quad \to \quad b^+ = \alpha(R_1 + i\,e\,R_2)$$

$$[R_i, R_j] = -i\,e\,l_B^2\,\varepsilon_{ij3}$$

→
$$\begin{aligned}[b, b^+] &= \alpha^2[R_1 - i\,e\,R_2, R_1 + i\,e\,R_2] \\ &= i\,e\,\alpha^2([R_1, R_2] - [R_2, R_1]) \\ &= 2\alpha^2 l_B^2 \\ &\equiv 1\end{aligned}$$

→ $$\alpha = \frac{1}{\sqrt{2}\,l_B}$$

$$b = \frac{1}{\sqrt{2}\,l_B}(R_1 - i\,e\,R_2) \qquad b^+ = \frac{1}{\sqrt{2}\,l_B}(R_1 + i\,e\,R_2)$$

$$b\,b^+ = \frac{1}{2\,l_B^2}\left(R_1^2 + R_2^2 + i\,e\,[R_1, R_2]\right)$$

$$b^+\,b = \frac{1}{2\,l_B^2}\left(R_1^2 + R_2^2 - i\,e\,[R_1, R_2]\right)$$

→ $$b\,b^+ + b^+\,b = \frac{1}{l_B^2}\boldsymbol{R}^2 = 2\,b^+\,b + 1$$

$$[v_i, R_j] = 0$$

→ $$[a, b] = [a, b^+] = 0$$

$$H = \hbar\omega_c\left(a^+\,a + \frac{1}{2}\right)$$

→ $$\dot{b} = \frac{1}{i\hbar}[b, H] = 0 = \dot{b}^+$$

Thus, the stationary states of the system can be written as $|\,n, m\rangle$ with

$$a^+\,a\,|\,n, m\rangle = n\,|\,n, m\rangle \quad \& \quad b^+\,b\,|\,n, m\rangle = m\,|\,n, m\rangle$$

where n is the label of the Landau level. The meaning of m is dicussed in 10.3._Symmetric-Gauge.pdf.

$$\boldsymbol{R}^2 = \left(r_i + \frac{e}{\omega_c}\,\varepsilon_{i\,j\,3}\,v_j\right)\left(r_i + \frac{e}{\omega_c}\,\varepsilon_{i\,k\,3}\,v_k\right)$$

$$= \boldsymbol{r}^2 + \frac{1}{\omega_c^2}\,\varepsilon_{i\,j\,3}\,\varepsilon_{i\,k\,3}\,v_j\,v_k + \frac{e}{\omega_c}\left(\varepsilon_{i\,k\,3}\,r_i\,v_k + \varepsilon_{i\,j\,3}\,v_j\,r_i\right)$$

$$= \boldsymbol{r}^2 + \frac{1}{\omega_c^2}\,\boldsymbol{v}^2 + \frac{e}{\omega_c}\left(\varepsilon_{i\,k\,3}\,r_i\,v_k + \varepsilon_{i\,j\,3}\,v_j\,r_i\right)$$

$$\left[r_i, v_j\right] = i\,\frac{\hbar}{M}\,\delta_{i\,j}$$

$$\rightarrow \qquad \varepsilon_{i\,j\,3}\,v_j\,r_i = \varepsilon_{i\,j\,3}\,r_i\,v_j$$

$$\therefore \qquad \boldsymbol{R}^2 = \boldsymbol{r}^2 + \frac{1}{\omega_c^2}\,\boldsymbol{v}^2 + \frac{2\,e}{\omega_c}\,(\boldsymbol{r}\times\boldsymbol{v})_3$$

The angular momentum about the gliding center is

$$\mathcal{L}_3 = M\,(\boldsymbol{\chi}\times\boldsymbol{v})_3 \;= M\,\varepsilon_{i\,j\,3}\,\chi_i\,v_j$$

$$\chi_i = -\frac{e}{\omega_c}\,\varepsilon_{i\,j\,3}\,v_j$$

$$\rightarrow \qquad \mathcal{L}_3 = -\frac{M\,e}{\omega_c}\,\varepsilon_{i\,j\,3}\,\varepsilon_{i\,k\,3}\,v_k\,v_j \;= -\frac{M\,e}{\omega_c}\,\boldsymbol{v}^2 \;= -2\,\frac{H\,e}{\omega_c} \;= -e\,\hbar\left(2\,a^+\,a + 1\right)$$

$$\therefore \qquad H = -e\,\frac{1}{2}\,\omega_c\,\mathcal{L}_3 = \frac{1}{2}\,\omega_c\,\left|\,\mathcal{L}_3\,\right|$$

i.e., the energy is entirely rotational, as expected.

The angular momentum with respect to the origin is defined as

$$L_3 = M\,(\boldsymbol{r}\times\boldsymbol{v})_3 \;= \frac{1}{2}\,e\,M\left(\omega_c\,\boldsymbol{R}^2 - \frac{1}{\omega_c}\,\boldsymbol{v}^2 - \omega_c\,\boldsymbol{r}^2\right)$$

$$= \frac{1}{2}\left(\frac{e\,B}{c}\,\boldsymbol{R}^2 - \frac{c\,M^2}{e\,B}\,\boldsymbol{v}^2 - \frac{e\,B}{c}\,\boldsymbol{r}^2\right) \qquad \left(\, l_B^2 = \frac{\hbar\,c}{|\,e\,|\;B}\,\right)$$

$$= \frac{1}{2}\left(e\,\hbar\left(2\,b^+\,b + 1\right) - e\,\hbar\left(2\,a^+\,a + 1\right) - \frac{e\,B}{c}\,\boldsymbol{r}^2\right)$$

$$= e\,\hbar\left(b^+\,b - a^+\,a\right) - \frac{e\,B}{2\,c}\,\boldsymbol{r}^2$$

$$= e\left(\hbar\left(b^+\,b - a^+\,a\right) - \frac{1}{2}\,M\,\omega_c\,\boldsymbol{r}^2\right)$$

In the symmetric gauge (see 10.3._SymmetricGauge.pdf), it reduces to

$$L_3 = e\,\hbar\left(b^+\,b - a^+\,a\right)$$