10.4. Landau Gauge

Landau gauge:

$$\boldsymbol{A} = B\,(-y,\,0,\,0)$$

$$\therefore \qquad \boldsymbol{D} = \left(\partial_x + i\,\frac{e\,B}{\hbar\,c}\,y\,,\;\partial_y\right)$$

$$= \left(\partial_x + i\,\frac{\mathbb{e}}{l_B^2}\,y\,,\;\partial_y\right) \qquad \left(\,l_B^2 = \frac{\hbar\,c}{|\,e\,|\;B}\,\right)$$

$$\boldsymbol{v} = \left(\frac{\hbar}{M\,i}\,\partial_x + \mathbb{e}\,\omega_c\,y\,,\;\frac{\hbar}{M\,i}\,\partial_y\right) \qquad \left(\,\omega_c = \frac{|\,e\,|\;B}{M\,c} = \frac{\hbar}{M\,l_B^2}\,\right)$$

$$\therefore \qquad \frac{\mathbb{e}}{\omega_c}\,\boldsymbol{v} = \left(-i\,\mathbb{e}\,l_B^2\,\partial_x + y\,,\;-i\,\mathbb{e}\,l_B^2\,\partial_y\right)$$

$$\rightarrow \qquad \boldsymbol{v}^2 = -\frac{\hbar^2}{M^2}\left\{\left(\partial_x + i\,\frac{\mathbb{e}}{l_B^2}\,y\right)^2 + \partial_y^2\right\}$$

$$= -\frac{\hbar^2}{M^2}\left(\partial_x^2 + \partial_y^2 - \frac{1}{l_B^4}\,y^2 + 2\,i\,\frac{\mathbb{e}}{l_B^2}\,y\,\partial_x\right)$$

$$\therefore \qquad \frac{1}{\omega_c^2}\,\boldsymbol{v}^2 = -l_B^4\left(\partial_x^2 + \partial_y^2\right) + y^2 - 2\,i\,\mathbb{e}\,l_B^2\,y\,\partial_x$$

$$\boldsymbol{R} = \boldsymbol{r} - \frac{\mathbb{e}}{\omega_c}\,\hat{\boldsymbol{z}}\times\boldsymbol{v}$$

$$= \left(x + \frac{\mathbb{e}}{\omega_c}\,v_2\,,\;y - \frac{\mathbb{e}}{\omega_c}\,v_1\right)$$

$$= \left(x - i\,\mathbb{e}\,l_B^2\,\partial_y\,,\;i\,\mathbb{e}\,l_B^2\,\partial_x\right)$$

$$\rightarrow \qquad \boldsymbol{R}^2 = \left(x - i\,\mathbb{e}\,l_B^2\,\partial_y\right)^2 - l_B^4\,\partial_x^2$$

$$= x^2 - l_B^4\left(\partial_x^2 + \partial_y^2\right) - 2\,i\,\mathbb{e}\,l_B^2\,x\,\partial_y$$

$$= x^2 - y^2 + \frac{1}{\omega_c^2}\,\boldsymbol{v}^2 - 2\,i\,\mathbb{e}\,l_B^2\left(x\,\partial_y - y\,\partial_x\right)$$

$$= x^2 - y^2 + \frac{1}{\omega_c^2}\,\boldsymbol{v}^2 + 2\,\mathbb{e}\,l_B^2\,\frac{L_3}{\hbar}$$

$$H = \frac{1}{2}\,M\,\boldsymbol{v}^2 = -\frac{\hbar^2}{2\,M}\left(\partial_x^2 + \partial_y^2 - \frac{1}{l_B^4}\,y^2 + 2\,i\,\frac{\mathbb{e}}{l_B^2}\,y\,\partial_x\right)$$

Since $\dfrac{\partial H}{\partial x} = 0$, we have

$$H\,\Psi = E\,\Psi$$

$$\rightarrow \qquad \Psi(x,\,y) = e^{i\,k\,x}\,\psi(y)$$

$$\& \qquad -\frac{\hbar^2}{2\,M}\left(-k^2 + \partial_y^2 - \frac{1}{l_B^4}\,y^2 - 2\,\frac{\mathbb{e}}{l_B^2}\,k\,y\right)\psi = E\,\psi$$

$$\left\{-\partial_y^2 + \frac{1}{l_B^4}\left(y + \mathbb{e}\,l_B^2\,k\right)^2\right\}\psi = \frac{2\,M\,E}{\hbar^2}\,\psi$$

$$\boldsymbol{R} = \left(x - i\,\mathbb{e}\,l_B^2\,\partial_y\,,\;-\mathbb{e}\,l_B^2\,k\right) \equiv (X,\,Y)$$

$\rightarrow \qquad \left\{ -\partial_y^2 + \frac{1}{l_B^4} (y - Y)^2 \right\} \psi = \frac{2ME}{\hbar^2} \psi$

Note: In this form, X is an operator but Y is just a number.

In dimensionless form

$$\left\{ -(l_B \partial_y)^2 + \left(\frac{y - Y}{l_B} \right)^2 \right\} \psi = \frac{2E}{\hbar \omega_c} \psi$$

Let

$$a = \alpha \left(\frac{y - Y}{l_B} + l_B \partial_y \right) \qquad \rightarrow \qquad a^+ = \alpha^* \left(\frac{y - Y}{l_B} - l_B \partial_y \right)$$

$\therefore \qquad [a, a^+] = \alpha\alpha^* \left(-[y, \partial_y] + [\partial_y, y] \right) = 2\alpha\alpha^* \equiv 1$

$\rightarrow \qquad a = \frac{1}{\sqrt{2}} \left(\frac{y - Y}{l_B} + l_B \partial_y \right) \qquad\qquad a^+ = \frac{1}{\sqrt{2}} \left(\frac{y - Y}{l_B} - l_B \partial_y \right)$

$$a\, a^+ = \frac{1}{2} \left\{ \left(\frac{y - Y}{l_B} \right)^2 - (l_B \partial_y)^2 - [y, \partial_y] \right\} = \frac{1}{2} \left\{ \left(\frac{y - Y}{l_B} \right)^2 - (l_B \partial_y)^2 + 1 \right\}$$

$$a^+ a = \frac{1}{2} \left\{ \left(\frac{y - Y}{l_B} \right)^2 - (l_B \partial_y)^2 + [y, \partial_y] \right\} = \frac{1}{2} \left\{ \left(\frac{y - Y}{l_B} \right)^2 - (l_B \partial_y)^2 - 1 \right\}$$

$\therefore \qquad a\, a^+ + a^+ a = \frac{2E}{\hbar \omega_c}$

$\rightarrow \qquad E = \hbar \omega_c \left(a^+ a + \frac{1}{2} \right)$

$$= \hbar \omega_c \left(n + \frac{1}{2} \right) \qquad\qquad n = 0, 1, 2, \ldots$$

which is the same as the symmetric gauge, as it must be.

Using

$$\partial_y e^{-\alpha y^2} = -2\alpha y\, e^{-\alpha y^2}$$

$$\partial_y^2 e^{-\alpha y^2} = (-2\alpha + 4\alpha^2 y^2)\, e^{-\alpha y^2}$$

we see that the solution to the ground state eq.

$$\left\{ -(l_B \partial_y)^2 + \left(\frac{y - Y}{l_B} \right)^2 \right\} \psi_0 = \psi_0$$

is

$$\alpha = \frac{1}{2 l_B^2}$$

$\& \qquad \psi_0 = C \exp\left\{ -\frac{1}{2} \left(\frac{y - Y}{l_B} \right)^2 \right\}$

$$\langle \psi_0 \mid \psi_0 \rangle = C^* C \int_{-\infty}^{\infty} dy \exp\left\{ -\left(\frac{y - Y}{l_B} \right)^2 \right\} = C^* C \sqrt{\pi}\, l_B = 1$$

$\rightarrow \qquad \psi_0 = \frac{1}{\sqrt{\pi^{1/2} l_B}} \exp\left\{ -\frac{1}{2} \left(\frac{y - Y}{l_B} \right)^2 \right\}$

$$\Psi_{k0}(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^{1/2} l_B}} e^{ikx} \exp\left\{ -\frac{1}{2} \left(\frac{y - Y}{l_B} \right)^2 \right\}$$

Note that

$$\langle k\,0 \mid k'\,0 \rangle = \int d^2 r\, \Psi^*_{k\,0}(\boldsymbol{r})\, \Psi_{k'\,0}(\boldsymbol{r})$$

$$= \int d\,x\, e^{i\,(-k+k')\,x} = 2\,\pi\,\delta(k-k')$$

Higher Landau levels are given by

$$\langle \boldsymbol{r} \mid k\,n \rangle = \Psi_{k\,n}(\boldsymbol{r}) = \left\langle \boldsymbol{r} \,\middle|\, \frac{a^{+n}}{\sqrt{n!}} \,\middle|\, k\,0 \right\rangle$$

$$= \frac{1}{\sqrt{2^n\,n!}} \left(\frac{y-Y}{l_B} - l_B\,\partial_y \right)^n \Psi_{k\,0}(\boldsymbol{r}) \qquad (Y = -\mathbb{e}\, l_B^2\, k)$$

which describes an oscillator in the y direction whose center is moving freely along the x-axis.

Consider a system of dimensions $L_x \times L_y$.

Imposing the periodic boundary condition on the plane wave makes

$$k = \frac{2\,\pi}{L_x}\, m \qquad m = 0,\ 1,\ 2,\ \ldots$$

so that the smallest separation between allowable values is $\Delta\,k = \dfrac{2\,\pi}{L_x}$.

This in turn makes

$$\Delta\,Y = \left| -\mathbb{e}\, l_B^2\, \Delta\,k \right| = l_B^2\, \frac{2\,\pi}{L_x}$$

Each oscillator thus runs within a strip parallel to the x-axis & of effective width $\Delta\,Y$. The area it occupies is

$$\Delta\,S = L_x\,\Delta\,Y = 2\,\pi\, l_B^2$$

which is the same as in the symmetric gauge.

The Noether current density is (see 10.3._SymmetricGauge.pdf)

$$\boldsymbol{J} = i\, \frac{e\,\hbar}{2\,M} [(\boldsymbol{D}\psi)^+\, \psi - \psi^+\, \boldsymbol{D}\psi]$$

$$D_x\, \Psi_{k\,n}(\boldsymbol{r}) = \left(\partial_x + i\, \frac{\mathbb{e}}{l_B^2}\, y \right) e^{i\,k\,x}\, \psi_n(y)$$

$$= i \left(k + \frac{\mathbb{e}}{l_B^2}\, y \right) \Psi_{k\,n}(\boldsymbol{r})$$

$$D_y\, \Psi_{k\,n}(\boldsymbol{r}) = e^{i\,k\,x}\, \partial_y\, \psi_n(y)$$

For $n = 0$ & switching to box normalization for the plane wave,

$$\Psi_{k\,0}(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^{1/2}\, l_B}}\, \frac{e^{i\,k\,x}}{\sqrt{L_x}}\, \exp\left\{ -\frac{1}{2} \left(\frac{y-Y}{l_B} \right)^2 \right\}$$

→ $$\partial_y\, \psi_0 = -\left(\frac{y-Y}{l_B} \right) \psi_0$$

∴ $$J_x = i\, \frac{e\,\hbar}{2\,M} \left\{ -i \left(k + \frac{\mathbb{e}}{l_B^2}\, y \right) - i \left(k + \frac{\mathbb{e}}{l_B^2}\, y \right) \right\} \Psi^*_{k\,0}\, \Psi_{k\,0}$$

$$= \frac{e\,\hbar}{M} \left(k + \frac{\mathbb{e}}{l_B^2}\, y \right) \Psi^*_{k\,0}\, \Psi_{k\,0}$$

$$= \frac{|\,e\,|\,\hbar}{M\, l_B^3 \sqrt{\pi}\, L_x} (-Y + y) \exp\left\{-\left(\frac{y - Y}{l_B}\right)^2\right\} \qquad (\, Y = -\,\mathbb{e}\, l_B^2\, k\,)$$

$$\rightarrow \qquad \langle J_x \rangle = \int_0^{L_x} d\,x \int_{-\infty}^{\infty} d\,y\, J_x \; = 0$$

$$J_y = i\, \frac{e\,\hbar}{2\,M} \left\{ -\left(\frac{y - Y}{l_B}\right) + \left(\frac{y - Y}{l_B}\right) \right\} \Psi_{k\,0}^{*}\, \Psi_{k\,0} \; = 0$$

That the average (macroscopic) linear current vanishes in all directions is to be expected for cyclotronic motion.