10.5. von Neumann Lattice

See 1.5._CoherentStatesAndVonNeumannLattice.pdf .

For $m=0$, the state $|\, n,\, 0\rangle$ in the symmetric gauge represents a cyclotron orbit centered at the origin. The size of this orbit is $2\pi\, l_B^2\,(n+1)$ (see 10.2._CyclotronMotion.pdf).

Reminder: n denotes the energy of the cyclotron motion & m denotes the distance of its center from the origin. Thus, m also denotes the difference between the intrinsic angular momentum $\mathcal{L}_3$ (pivot point at orbit center) & the angular momentum L_3 (pivot point at origin).

As described in §1.5, the eigenstates of H can be taken as the states obtained by displacing $\langle \boldsymbol{r} \,|\, n,\, 0\rangle$ to the points of a lattice of unit cell size $2\pi\, l_B^2\,(n+1)$.

For a square lattice, the lattice constant is $\sqrt{2\pi\, l_B^2\,(n+1)}$ so that the guiding centers (lattice points) are given by

$$\boldsymbol{R}_{jk} = \left(X_j,\, Y_k\right) = \sqrt{2\pi\,(n+1)}\;\; l_B\,(j,\, k) \qquad\qquad j,\, k = 0,\, \pm 1,\, \pm 2,\, \ldots$$

Since (see 10.3._SymmetricGauge.pdf)

$$\boldsymbol{R}^2 = 2\, l_B^2 \left(b^+\, b + \frac{1}{2}\right)$$

the displacement operator is given by (see §1.5)

$$D(\beta) = e^{\beta\, b^+ - \beta^*\, b} \text{ with} \qquad\qquad \beta = \beta_R + i\,\beta_I = \frac{1}{\sqrt{2}\; l_B}\left(R_x - i\,\mathbb{e}\, R_y\right)$$

The reason for the $-\mathbb{e}$ factor will be clear later on.

$$e^{A+B} = e^A\, e^B\, e^{-[A,B]/2}$$

$$\rightarrow \qquad D(\beta) = e^{-\beta^*\, b}\, e^{\beta\, b^+}\, e^{|\beta|^2/2}$$

$$= e^{\beta\, b^+}\, e^{-\beta^*\, b}\, e^{-|\beta|^2/2}$$

The von Neumann lattice is thus given by

$$\beta_{jk} = \sqrt{\pi(n+1)}\;\,(j + i\,k) \quad \sim \quad \sqrt{\pi(n+1)}\;\,(j,\, k)$$

$$\left|\; n,\, \boldsymbol{R}_{jk}\right\rangle = \left|\; n,\; X_j\,,\, Y_k\;\right\rangle = \left|\; n,\, \beta_{jk}\;\right\rangle$$

$$= D\left(\beta_{jk}\right)\,\left|\; n,\, 0\;\right\rangle$$

$$= e^{\beta_{jk}\, b^+ - \beta_{jk}^*\, b}\,\left|\; n,\, 0\;\right\rangle$$

$$= e^{-|\beta_{jk}|^2/2}\, e^{\beta_{jk}\, b^+}\, e^{-\beta_{jk}^*\, b}\,\left|\; n,\, 0\;\right\rangle$$

$$= e^{-|\beta_{jk}|^2/2}\, e^{\beta_{jk}\, b^+}\,\left|\; n,\, 0\;\right\rangle \qquad\qquad (\, b \,|\, 0\rangle = 0\,)$$

$$= e^{-\pi(n+1)\,(j^2+k^2)/2}\, e^{\sqrt{\pi(n+1)}\;\,(j+i\,k)\, b^+}\,\left|\; n,\, 0\;\right\rangle$$

From §1.5,

$$\langle \beta \,|\, \alpha\rangle = \exp\left(-\frac{1}{2}\,(\beta^*\,\beta + \alpha^*\,\alpha) \;+\; \beta^*\,\alpha\right)$$

$$(\beta - \alpha)^*\,(\beta - \alpha) = \beta^*\,\beta + \alpha^*\,\alpha - \beta^*\,\alpha - \alpha^*\,\beta$$

$$\rightarrow \qquad \langle \beta \,|\, \alpha\rangle = \exp\left(-\frac{1}{2}\,(\beta - \alpha)^*\,(\beta - \alpha) + \frac{1}{2}\,(\beta^*\,\alpha - \alpha^*\,\beta)\right)$$

$$\therefore \qquad \left\langle n,\, \boldsymbol{R}_{j'k'} \,\middle|\, n',\, \boldsymbol{R}_{jk}\right\rangle = \delta_{n\,n'} \exp\left(-\frac{1}{2}\,\pi\,(n+1)\left[\,(j'-j)^2 + (k'-k)^2 \;-2\,i\,(j'\,k - k'\,j)\,\right]\right)$$

From §10.3, we have

$$b = \frac{1}{2\sqrt{2}}\left(\tilde{z}^* + 4\frac{\partial}{\partial \tilde{z}}\right) \qquad b^+ = \frac{1}{2\sqrt{2}}\left(\tilde{z} - 4\frac{\partial}{\partial \tilde{z}^*}\right)$$

& $$\Psi_{00}(\boldsymbol{r}) = \langle \boldsymbol{r} \mid 0, 0\rangle = \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{4}\tilde{z}\,\tilde{z}^*\right) \qquad (\tilde{z} = \tilde{r}\, e^{i\,\mathrm{e}\,\theta})$$

$$= \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{4}\tilde{r}^2\right)$$

$$\left\langle \boldsymbol{r} \mid b^+ \mid 0, 0\right\rangle = \frac{1}{2\sqrt{2}}\left(\tilde{z} - 4\frac{\partial}{\partial \tilde{z}^*}\right)\Psi_{00}(\boldsymbol{r}) = \frac{\tilde{z}}{\sqrt{2}}\Psi_{00}(\boldsymbol{r})$$

$$\left\langle \boldsymbol{r} \mid b^{+2} \mid 0, 0\right\rangle = \left(\frac{1}{2\sqrt{2}}\left(\tilde{z} - 4\frac{\partial}{\partial \tilde{z}^*}\right)\right)^2\Psi_{00}(\boldsymbol{r}) = \left(\frac{\tilde{z}}{\sqrt{2}}\right)^2\Psi_{00}(\boldsymbol{r})$$

→ $$\left\langle \boldsymbol{r} \mid b^{+n} \mid 0, 0\right\rangle = \left(\frac{\tilde{z}}{\sqrt{2}}\right)^n\Psi_{00}(\boldsymbol{r})$$

$$\left\langle \boldsymbol{r} \mid e^{\alpha b^+} \mid 0, 0\right\rangle = e^{\alpha \tilde{z}/\sqrt{2}}\,\Psi_{00}(\boldsymbol{r})$$

Setting

$$\tilde{\boldsymbol{R}} = \frac{1}{l_B}\boldsymbol{R} = \frac{1}{l_B}(X, Y) = (\tilde{X}, \tilde{Y}) \qquad \rightarrow \qquad \beta = \frac{1}{\sqrt{2}}(\tilde{R}_x - i\,\mathrm{e}\,\tilde{R}_y)$$

we have

$$\Psi_0(\boldsymbol{r};\, \boldsymbol{R}_{jk}) = \langle \boldsymbol{r} \mid 0, \boldsymbol{R}_{jk}\rangle$$

$$= \left\langle \boldsymbol{r} \mid e^{-|\beta_{jk}|^2/2}\, e^{\beta_{jk} b^+} \mid 0, 0\right\rangle$$

$$= \exp\left(-\frac{1}{2}|\beta_{jk}|^2 + \frac{1}{\sqrt{2}}\beta_{jk}\tilde{z}\right)\Psi_{00}(\boldsymbol{r})$$

$$= \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{2}|\beta_{jk}|^2 + \frac{1}{\sqrt{2}}\beta_{jk}\tilde{r}\, e^{i\,\mathrm{e}\,\theta} - \frac{1}{4}\tilde{r}^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{4}\left(\tilde{r}^2 + 2|\beta_{jk}|^2 - 2\sqrt{2}\,\beta_{jk}\tilde{r}\, e^{i\,\mathrm{e}\,\theta}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{4}\left[\tilde{r}^2 + \tilde{\boldsymbol{R}}_{jk}^2 - 2(\tilde{X}_j - i\,\mathrm{e}\,\tilde{Y}_k)\tilde{r}\, e^{i\,\mathrm{e}\,\theta}\right]\right)$$

$$(\tilde{X}_j - i\,\mathrm{e}\,\tilde{Y}_k)\tilde{r}\, e^{i\,\mathrm{e}\,\theta} = (\tilde{X}_j - i\,\mathrm{e}\,\tilde{Y}_k)(\tilde{x} + i\,\mathrm{e}\,\tilde{y}) = \tilde{X}_j\tilde{x} + \tilde{Y}_k\tilde{y} + i\,\mathrm{e}(\tilde{X}_j\tilde{y} - \tilde{Y}_k\tilde{x})$$

$$\tilde{r}^2 + \tilde{\boldsymbol{R}}_{jk}^2 - 2(\tilde{X}_j - i\,\mathrm{e}\,\tilde{Y}_k)\tilde{r}\, e^{i\,\mathrm{e}\,\theta} = \tilde{x}^2 + \tilde{y}^2 + \tilde{X}_j^2 + \tilde{Y}_k^2 - 2\left[\tilde{X}_j\tilde{x} + \tilde{Y}_k\tilde{y} + i\,\mathrm{e}(\tilde{X}_j\tilde{y} - \tilde{Y}_k\tilde{x})\right]$$

$$= (\tilde{x} - \tilde{X}_j)^2 + (\tilde{y} - \tilde{Y}_k)^2 + 2\,i\,\mathrm{e}(\tilde{x}\tilde{Y}_k - \tilde{y}\tilde{X}_j)$$

$$= (\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{R}}_{jk})^2 + 2\,i\,\mathrm{e}(\tilde{\boldsymbol{r}} \times \tilde{\boldsymbol{R}}_{jk})_3$$

∴ $$\Psi_0(\boldsymbol{r};\, \boldsymbol{R}_{jk}) = \frac{1}{\sqrt{2\pi}\, l_B}\exp\left(-\frac{1}{4}\left[(\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{R}}_{jk})^2 + 2\,i\,\mathrm{e}(\tilde{\boldsymbol{r}} \times \tilde{\boldsymbol{R}}_{jk})_3\right]\right)$$

The factor $(\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{R}}_{jk})^2$ is to be expected for moving the orbit center to $\tilde{\boldsymbol{R}}_{jk}$.

It is also why we've chosen

$$\beta = \tilde{R}_x - i\,\mathrm{e}\,\tilde{R}_y$$

The reason that we can assign either $\tilde{R}_x - i\,\tilde{R}_y$ or $\tilde{R}_x + i\,\tilde{R}_y$ to β can be traced to the freedom of switching $b \leftrightarrow b^{+}$.

The phase factor $\exp\left[-i\,e\left(\tilde{\boldsymbol{r}}\times\tilde{\boldsymbol{R}}_{jk}\right)_3\right]$ is akin to the factor $e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}}$ in the Bloch's theorem.